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# The Generalized Waring Distribution. Part I†

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(*Retired*)*

Summary

The Generalized Waring Distribution is the hypergeometric distribution whose generating function is given by $CF(a, k, \rho+a+k, A)$, $a \geqslant 0$, $k \geqslant 0$, $\rho > 0$, $C = \rho_{[k]}/(\rho+a)_{[k]}$. For certain values of the parameters $a, k$ it has extremely long tails; indeed all the moments can be infinite. (This need not be the case; where the first four moments are finite, it has been found useful in dealing with accident distributions.)

In Part I various cases are distinguished, corresponding to special values of the three parameters. General formulae for the factorial moments, also $\beta_1, \beta_2$, are given as well as the forms these take in special cases. The continuous analogue of the discrete distribution is defined. In general, it is Pearson's Type VI though Types III, IV and V can occur in particular cases. A table of $\beta_1, \beta_2$ is given and discussed for all combinations of the values $\rho = 8, 16, 24, 32, \infty, q_a = a/(a+\rho)$, $q_k = k/(k+\rho) = \frac{1}{9}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1$. The mode of the distribution is obtained and its properties discussed.

*Keywords*: GENERALIZED WARING DISTRIBUTION (G.W.D.); LONG TAILS; SPECIAL CASES; CONTINUOUS ANALOGUE (USUALLY PEARSON'S TYPE VI); MODE

## 1. Introduction

In my presidential address to the Royal Statistical Society (Irwin, 1963) (particularly in Section 2 of Appendix II), I discussed the use of inverse factorial series as frequency distributions. The parts of the field which are still unexplored must be very wide. However, I described rather fully what has come to be called the simple Waring distribution and gave several examples of its application, as well as extracts from its tabulation on an electronic computer. I also gave some of the properties of the Generalized Waring Distribution. I used the form of its generating function which may be written

$$\frac{(x-a)_{[k]}}{x_{[k]}} F\{a, k, x+k, A\}, \quad a \geqslant 0, \quad k \geqslant 0, \quad x \geqslant a, \tag{1.1}‡$$

where $F$ is the hypergeometric series, $A$ the generating symbol and $x_{[k]}$ denotes $x(x+1)\ldots(x+k-1)$.

However it is, in general, more convenient to put $x-a = \rho$ and write (1.1)

$$\frac{\Gamma(\rho+a)\,\Gamma(\rho+k)}{\Gamma(\rho)\,(\rho+a+k)}\left\{1+\frac{akA}{(\rho+a+k)}+\frac{a(a+1)\,k(k+1)}{(\rho+a+k)(\rho+a+k+1)}\frac{A^2}{2!} + \ldots\frac{a_{[r]}k_{[r]}}{(\rho+a+k)_{[r]}}\frac{A^r}{r!}+\ldots\right\}. \tag{1.2}$$

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† Parts II and III will appear in Parts 2 and 3, respectively, of the *Journal*, Vol. 138 (1975).

‡ In a previous paper (Irwin, 1968), I took $a > 0$, $k > 0$, $x > a$; here (1.1) seems preferable in view of the discussion in Section 2 below.

When $k = 1$, (1.2) takes the simple form

$$\frac{\rho}{\rho+a}\left\{1+\frac{aA}{(\rho+a+1)}+\frac{a(a+1)A^2}{(\rho+a+1)(\rho+a+2)}+\ldots+\frac{a_{[r]}A^r}{(\rho+a+1)_{[r]}}+\ldots\right\}, \tag{1.3}$$

which is the simple Waring distribution. The particular case of this when $a = 1$ is called the Yule distribution, because Yule was the first to use a distribution of this kind in his study of the distribution of genera size, according to the number of species per genus (Yule, 1924).

Because of the analogy of (1.3) with a geometric progression, it is convenient to write $q = a/(a+\rho)$ in (1.3). In the preparation of the paper to which reference has been made, the distributions (1.3) were tabulated for $q = 0{\cdot}1\,(0{\cdot}1)\,0{\cdot}9$, $(x-a) = \rho = 1\,(1)\,10$ and also for $\rho = \infty$, when the distributions become geometric progressions. (However, only the values for $q = 0{\cdot}1, 0{\cdot}5, 0{\cdot}9$ were published.) All the distributions proved to be J-shaped. For low values of $q$, their tails are long when $\rho = 1$ and decrease in length with increasing $\rho$. The tails also increase in length with increasing $q$ for fixed $\rho$. In general the tails are always long; for example, when $q = 0{\cdot}5$ and $\rho = 1$, it takes about 140 terms to reach an individual frequency of 0·00005 and even the geometric progression takes more than 17. When $q = 0{\cdot}9$ and $\rho = 1$, it takes 400 terms to reach 0·00005 (with 2·2 per cent of the total frequency beyond this point) and even the geometric progression needs about 80.

The series in (1.3) is actually summable to $n$ terms. This was not realized when the paper was published though Yule (1924, p. 38) has noted this for the particular case when $a = 1$. In fact, if $f_s$ is the $s$th frequency, so that

$f_r = (x-a)\,a_{[r-1]}\,|\,x_{[r]}$, the tail frequency beyond $f_r$ is:

$$\begin{aligned}\sum_{s=r+1}^{\infty} f_s &= \frac{(x-a)\,a_{[r]}}{x_{[r+1]}}\left\{1+\frac{(a+r)}{(x+r+1)}+\frac{(a+r)(a+r+1)}{(x+r+1)(x+r+2)}+\ldots\right\}\\
&= \frac{(x-a)\,a_{[r]}}{x_{[r]}}\left\{\frac{1}{x+r}+\frac{(a+r)}{(x+r)(x+r+1)}+\frac{(a+r)(a+r+1)}{(x+r)(x+r+1)(x+r+2)}+\ldots\right\}\\
&= \frac{(x-a)\,a_{[r]}}{x_{[r]}}\{(x+r)-(a+r)\}^{-1}\\
&= \frac{a_{[r]}}{x_{[r]}}\\
&= \frac{a_{[r]}}{(a+\rho)_{[r]}}.\end{aligned} \tag{1.4}$$

Thus

$$\sum_{s=r+1}^{\infty} f_s = (x-a)^{-1} f_{r+1}(a+\rho+r) = (\rho+a+r)\,f_{r+1}/\rho. \tag{1.4 bis}$$

No such simple expression has been found for the tail of (1.2).

It was encountering frequency distributions with very long tails, actually occurring in nature, which led to the investigation of the distribution (1.3). Indeed it was found to give a good fit to the extremely long-tailed distribution of the numbers of filarial worms on 2600 mites. Arguments of a mathematico-biological nature were put forward which would lead to this form of distribution as a theoretical model (Irwin, 1963).

However, as has been mentioned, all the distributions (1.3) are J-shaped. It was the search for a discrete distribution which could have a mode and at the same time a very long tail, which led to the investigation of the more general form (1.2) in greater detail. Once this had been undertaken, these Generalized Waring Distributions were found to have a number of interesting and previously unsuspected properties. Some of these were discovered by tabulation on an electronic computer, for which I am indebted to Dr David Hill. In Parts I and II of the paper, the properties of the distribution are considered in some detail. Part I deals first with special limiting cases for particular values of the parameters; then with the general properties of the distribution—moments, shape, existence and position of the mode, etc. Part II examines these properties in the light of the computer tabulations, and also contains a detailed discussion of the percentage points and the way in which these vary with the values of the parameters. Examples of fitting the distribution are also given in Part II. It will be found that, though the distribution can have very long tails for certain values of the parameters, this is not the case for all values of the parameters.

In fact, since the distribution has a close relation to the continuous distribution which has the same ratio of slope to ordinate at the mid-point of each rectangle of its histogram—and the actual distribution is hypergeometric—the corresponding continuous distribution will be a Pearson frequency distribution. This analogy is discussed in Part III; it is shown that the corresponding continuous distribution (which we term the "continuous analogue") is Pearson's Type VI, or one of its limiting forms Type III, Type V or normal, with certain exceptions, which will be indicated, where it is of Type IV. However, the corresponding Pearson Type VI will not necessarily have four finite moments, and we are thus led to consider cases which Karl Pearson (at the time not unreasonably) would have excluded as heterotypic, but which might nevertheless arise as theoretical models of natural phenomena.

In another paper (Irwin, 1968) it has been shown that the Generalized Waring Distribution† can provide a theoretical model for accident distributions, by the use of which it is possible to allow separately for accident proneness and accident liability.

## 2. Special Cases

The series is always convergent. In certain limiting cases the series is non-uniformly convergent. This means that every term tends to zero, but the limit of the sum function is still unity. In this case we shall say that the distribution has an "infinitely long tail".

We have already discussed the special case when $k = 1$ (Irwin, 1963). The distribution then reduces to the simple Waring distribution. The same is of course true when $a = 1$. If $a = 1$ and $k = 1$, the distribution is the Yule distribution.

It is convenient to write

$$q_a = \frac{a}{(a+\rho)}, \quad p_a = \frac{\rho}{(a+\rho)}, \quad q_k = \frac{k}{(k+\rho)}, \quad p_k = \frac{\rho}{(k+\rho)}. \tag{2.1}$$

The following limiting cases arise:

† For the sake of brevity, I shall refer to this below as the "G.W. distribution" or "G.W.D."

(i) If $\rho \to 0$, the distribution has an infinitely long tail for all values of $a, k > 0$. In this case $q_a \to 1$ and $q_k \to 1$. The mean and all other moments are infinite.

(ii) If $\rho \to \infty$, $a \to \infty$ and $q_a$ $(0 < q_a < 1)$ is constant, $k \neq 0$ remaining finite, the limiting form of the distribution is the negative binomial

$$\left(\frac{1}{p_a} - \frac{q_a A}{p_a}\right)^{-k}.$$

When, also, $k \to \infty$ and $q_a \to 0$, $kq_a$ remaining finite, we obtain the Poisson distribution. The same form is reached if $k$ and $a$ are interchanged.

(iii) If $\rho \to \infty$, first then $a \to \infty$, $k \to 0$ (which implies $q_k \to 0$), or $\rho \to \infty$ first, $a \to 0$, $k \to \infty$ $(q_a \to 0)$, or $\rho \to \infty$ first, $a \to 0$ and $k \to 0$ ($q_k \to 0$ and $q_a \to 0$) all the frequency is concentrated in the first term.†

(iv) If $\rho > 0$ is finite, $k \geqslant 0$ is finite and $a \to \infty$ $(q_a \to 1)$, we reach a distribution with an infinitely long tail. The same is true if $a$ and $k$ are interchanged. In this case $\beta_1$ for $\rho > 3$, and $\beta_2$ for $\rho > 4$ have finite limits, depending on $\rho$ (see Sections 3 and 4). This point is further discussed below.

(v) If $\rho$ is finite, $a \to \infty$ and $k \to \infty$ $(q_a \to 1, q_k \to 1)$, we again reach a distribution with an infinitely long tail. $\beta_1$ for $\rho > 3$ and $\beta_2$ for $\rho > 4$ again tend to finite limits depending on $\rho$ (see Sections 3 and 4).

(vi) (a) If $a \to \infty$, $k \to \infty$ $(q_a \to 1, q_k \to 1)$ and then $\rho \to \infty$, we again have a distribution with an infinitely long tail, but $\beta_1 \to 0$, $\beta_2 \to 3$, their normal values (see Sections 3 and 4).

(b) If $\rho \to \infty$ first and then $a \to \infty$, $k \to \infty, q_a, q_k$ remaining fixed and $< 1$, we still have a curve with an infinitely long tail, and it is still true that $\beta_1 \to 0$ and $\beta_2 \to 3$ (see Sections 3 and 4).

## 3. MOMENTS

The $r$th factorial moment of the G.W.D. is given by

$$\mu_{[r]} = \frac{a_{[r]} k_{[r]}}{(\rho-1)(\rho-2)\ldots(\rho-r)}. \tag{3.1}$$

From (3.1) it follows immediately that all $r$th moments (e.g. ordinary moments about any origin, central moments as well as factorial moments) are infinite if $\rho \leqslant r$.

Moments about any origin, including central moments, can be obtained from (3.1) by the usual transformation formulae. In particular the mean is given by

$$\mu_1 = \frac{ak}{(\rho-1)}, \quad \rho > 1, \tag{3.2}$$

while the variance

$$\sigma^2 = \mu_2 = \frac{ka(\rho+a-1)(\rho+k-1)}{(\rho-1)^2(\rho-2)}, \quad \rho > 2. \tag{3.3}$$

† If $a$ or $k \to 0$ first, the result still holds good; but if $a$ or $k \to \infty$ first, every term, including the first, tends to zero. Since the sum function is unity, the distribution then has an infinitely long tail. In the former case all cumulants are zero and $\beta_1$, $\beta_2$ are indeterminate. In the latter case $\beta_1 \to 0$, $\beta_2 \to \infty$, $\beta_2 - \beta_1 - 1 \to \infty$. This shows that numerical cases approximating to these conditions need very careful consideration.

The coefficient of variation $v = \sigma/\mu_1$ is given by

$$(ak)^{-\frac{1}{2}}\{(\rho+a-1)(\rho+k-1)|(\rho-2)\}^{\frac{1}{2}}, \quad \rho>2. \tag{3.4}$$

The mean is infinite if $\rho \leqslant 1$ and the variance is infinite if $\rho \leqslant 2$.

The values of $\beta_1, \beta_2$ are

$$\beta_1 = \frac{(\rho+2a-1)^2(\rho+2k-1)^2(\rho-2)}{ak(\rho+a-1)(\rho+k-1)(\rho-3)^2}, \quad \rho>3, \tag{3.5}$$

$$\left.\begin{aligned} \beta_2 &= \frac{(\rho-2)L}{ak(\rho+a-1)(\rho+k-1)(\rho-3)(\rho-4)}, \quad \rho>4, \\ \text{where} \quad L &= (\rho-1)^4+\{3ak+6(a+k)+1\}(\rho-1)^3 \\ &\quad +\{3ak(a+k)+6(a^2+k^2+3ak)\}(\rho-1)^2 \\ &\quad +\{3a^2k^2+18ak(a+k)\}(\rho-1)+18a^2k^2. \end{aligned}\right\} \tag{3.6}$$

## 4. VALUES OF $\beta_1$, $\beta_2$

Values of $\beta_1, \beta_2$ have been calculated over a suitable range and are given in Table 1. Before discussing the table, the values of $\beta_1, \beta_2$ in the limiting cases mentioned above (Section 2) will be considered.

In case (i), the formulae (3.5) and (3.6) do not hold, and $\beta_1, \beta_2$ have no relevance. If they can be regarded as existing, they are indeterminate.

In case (iii), the mean is 0 or 1 according as the first frequency is at 0 or 1, and all other cumulants are zero. Formulae (3.5) and (3.6) give $\beta_1 \to \infty$ and $\beta_2 \to \infty$. If we let $\rho \to \infty$ first and then $a$ or $k \to 0$, we can verify that the relation $\beta_2 - \beta_1 - 1 \geqslant 0$ is satisfied, as must always be the case (Pearson, 1916).

Case (ii) is the negative binomial. The continuous analogue is Pearson's Type VI. Here we have

$$\beta_1 = (1+q_a)^2/kq_a, \quad \beta_2 = \beta_1+3+2/k, \tag{4.1}$$

which are familiar formulae. We note that if $a \to \infty$ or $q_a \to 1$ ($a$ and $k$ may of course be interchanged),

$$\beta_1 = 4/k, \quad \beta_2 - 3 = 6/k \quad \text{so that} \quad 2\beta_2 - 3\beta_1 - 6 = 0.$$

This is the condition for Pearson's Type III (the Gamma Distribution). When $q_a \to 1$, the negative binomial has an "infinitely long tail", as it has here been defined. If as is usual in actual physical or biological problems involving discrete distributions, the variate *must* take the values 0, 1, 2, 3 etc., then we have the infinitely long tail. If however the variate values can be 0, $\lambda$, $2\lambda$, $3\lambda$ etc., and $\lambda$ is at our choice, we can make $\lambda \to 0$ in such a way that $\Sigma^2 = \lambda^2\sigma^2 = \lambda^2 kq_a/p_a^2$ remains finite $= c^2k$, say. As is well known, we have in the limit a continuous frequency distribution of the Pearson Type III form, which can be written

$$\frac{1}{\Gamma(k)} e^{-x/c}\left(\frac{x}{c}\right)^{k-1}\frac{dx}{c}, \quad 0<x<\infty. \tag{4.2}$$

Thus, if we are at liberty to choose the scale, we can reach in this case a continuous distribution as a limiting form of the G.W.D., in which the tail is not infinitely long in any acceptable sense of the term.

It can now be seen that for any discrete distribution in which the variate has a natural metric and a finite variance, there are two quite distinct ways of measuring length of tail. The first is by the number of terms necessary to reach a given individual small frequency (say 0·00005) or alternatively some selected percentage point (say 0·01). The second is by the number of multiples of the S.D. required for the same purposes. Which is preferable will depend upon the aim in view. This point will be taken up again later.

Case (iv) is analogous to case (ii) with $q_a \to 1$. Here $\rho$ is finite and $a \to \infty$, and $k \geqslant 0$ is finite, or vice versa. Thus, either $q_a \to 1$, $q_k > 0$ or vice versa.

In this case if $k \to \infty$, and $a$, $\rho$ are both finite

$$\sigma^2 \sim \frac{k^2 a(\rho+a-1)}{(\rho-1)^2(\rho-2)} \to \infty, \tag{4.3}$$

$$\left.\begin{aligned} \beta_1 &= \frac{4(\rho-2)(\rho+2a-1)^2}{a(\rho+a-1)(\rho-3)^2}, \\ \beta_2 &= \frac{3(\rho-2)}{a(\rho+a-1)(\rho-3)(\rho-4)}\{(a+2)(\rho-1)^2+a(a+6)(\rho-1)+6a^2\}. \end{aligned}\right\} \tag{4.4}$$

If the scale is at our choice we can take the variate values as $r/k$ ($r = 0, 1, 2, \ldots$) in which case we reach (when $k \to \infty$) a continuous distribution with variance $\Sigma^2$, where

$$\Sigma^2 = \frac{a(\rho+a-1)}{(\rho-1)^2(\rho-2)}. \tag{4.5}$$

It will be shown in Part III that the continuous analogue of the G.W. distribution is in general Pearson's Type VI.

[We define the continuous analogue to be the distribution obtained by the slope ordinate method; i.e. by equating $(1/y)(dy/dx)$ to the ratio of slope to ordinate in the histogram of the discrete distribution, which is in fact a particular type of hypergeometric distribution. It is argued in Part II that this is the correct method to use. If we do so, we find that the continuous analogue is always Type VI, for $\rho > (2\sqrt{2}+3)$; or one of its limiting cases, Type III, Type V or normal. If changes of scale are permissible, the limiting forms for special values of $a$, $k$, $\rho$ can also be regarded as limiting forms of the G. W. distribution itself.

However, the slope-ordinate method does not, in general, give the same Pearson distribution as equating the $\beta_1, \beta_2$ of the Pearson distribution to the $\beta_1, \beta_2$ of the hypergeometric. These are two reasons why the former method is preferable.

(i) It is applicable to cases where any or all of the first four moments are infinite. This can be the case with exceptionally long-tailed distributions, which do occasionally occur in biological material. In Yule's example—the distribution of size of genera—even the mean is infinite (Yule, 1924). In my own example of the distribution of filarial worms per mite (Irwin, 1963) the mean is finite but the variance (since $\rho = 1{\cdot}85$) is infinite.

(ii) The former method, but not the latter, yields a Pearson Type VI distribution in which $q_1, q_2$, in Pearson's notation for the distribution, satisfy the relation $\rho = q_1 - q_2 - 1$. (The curve can be written $y = Cx^{q_2}(x+a)^{-q_1}$, $0 \leqslant x < \infty$.)

Thus, in both the Waring distribution and its continuous analogue the successive moments become infinite for the same values of $\rho$.

Assuming the first method is used, the continuous analogue does not in general have the same $\beta_1, \beta_2$ as the hypergeometric. Since the continuous analogue is Type VI (or exceptionally Type IV) it must always have $2\beta_2 - 3\beta_1 - 6 > 0$. In the G. W. distribution however, when $\rho$ and $a$ are fixed and $k$ increases (here $a$ and $k$ may be interchanged), $2\beta_2 - 3\beta_1 - 6$ is positive for low values of $k$ and negative for high values, changing sign at some value of $\rho = \rho_0(a, k)$. When $\rho \to \infty, k \to \infty, 0 < q_k < 1$ (Case (ii)), the G. W. distribution becomes a negative binomial. Here $2\beta_2 - 3\beta_1 - 6 = -p_k^2/aq_k$, whereas the continuous analogue, given by the slope ordinate method gives $2\beta_2 - 3\beta_1 - 6 < 0$. It follows that if we fitted by equating $\beta_1, \beta_2$ of curve and negative binomial, we should obtain Type I and not Type VI. This seems illogical for a curve which starts at zero and has an unlimited tail to the right.

In his first Royal Society paper on "Skew Variation in Homogeneous Material", Karl Pearson (1895) did at first obtain his first four main types by the slope/ordinate method. However, in the same paper he then expressed the constants of the curves in terms of $\beta_1 = \mu_3^2/\mu_2^3$, $\beta_2 = \mu_4/\mu_2^2$. (In the earlier part of the paper he put his differential equation in the form $(1/y)(dy/dx) = -x/(\beta_1 + \beta_2 x + \beta_3 x^2)$ where $\beta_1, \beta_2, \beta_3$ are not the moment ratios; the change of notation might perhaps confuse the unwary reader.) He did not explicitly state, though he certainly would have recognized, that equating the $\beta_1, \beta_2$ of the curve and hypergeometric did not give the same answer as the slope-to-ordinate method. Since his main object was to graduate observed data, it was natural at the time that he should equate the observed $\beta_1, \beta_2$ to their theoretical values.

(For $0 < \rho < (2\sqrt{2}+3) = 5{\cdot}828$ there are certain exceptions when the continuous analogue is of Type IV or Type V. These exceptions will be further discussed in Part III.)

This distribution, in this case reduces to (see Part III)

$$\frac{\Gamma(\rho+a)}{\Gamma(\rho)\,\Gamma(a)}\,\xi^{a-1}(1+\xi)^{-(\rho+a)}\,d\xi \tag{4.6}$$

with variance given by (4.5). Assuming the change of scale to be permissible, it can be regarded as a limiting form of the G.W. distribution.

Case (v) is similar. We find

$$\left.\begin{aligned} \sigma^2 &\sim \frac{a^2 k^2}{(\rho-1)^2(\rho-2)} \to \infty, \\ \beta_1 &= \frac{16(\rho-2)}{(\rho-3)^2}, \\ \beta_2 &= \frac{3(\rho-2)(\rho+5)}{(\rho-3)(\rho-4)}. \end{aligned}\right\} \tag{4.7}$$

If the scale is at our disposal, we can suppose the variate values to be

$$r/ak \; (r = 0, 1, 2, 3, \ldots).$$

When $a \to \infty$, $k \to \infty$ we reach a continuous distribution in the limit with variance $1/(\rho-1)^2(\rho-2)$ and $\beta_1, \beta_2$ given by (4.7). On putting $\xi = 1/aX$ in (4.6) and letting $a \to \infty$, its equation is found to be

$$\frac{1}{\Gamma(\rho)}\,e^{-1/X}\,X^{-(\rho+1)}\,dX, \quad 0 \leqslant X < \infty, \tag{4.8}$$

that is Pearson's Type V.

The values of $\beta_1, \beta_2$ in (4.7) agree with those found by Karl Pearson (1901) for Type V, on putting $\rho+1 = \rho$.

On eliminating $\rho$ we obtain Karl Pearson's cubic

$$\beta_1(\beta_2+3)^2-4(2\beta_2-3\beta_1-6)(4\beta_2-3\beta_1) = 0. \tag{4.9}$$

In case (vi) (a), by making $\rho \to \infty$ in (4.8) we find $\beta_1 \to 0$, $\beta_2 \to 3$, their normal values. It is well known that Type V tends to the normal form as $\rho \to \infty$. On the other hand, in case (vi) (b) we still find $\beta_1 \to 0$, $\beta_2 \to 3$, but

$$\sigma^2 \sim (\rho q_a q_k / p_a^2\, p_k^2) = c^2, \quad \text{say}. \tag{4.10}$$

If we can change the scale, so that our variate values are $r/c$ ($r = 0, 1, 2, 3$ etc.) and then make $c \to \infty$, we still reach the normal distribution $\sqrt{(2\pi)^{-1}}\, e^{-\frac{1}{2}X^2}\, dx$. Under these conditions, the normal distribution can be a limiting form of the G.W. distribution.

We are now in a position to discuss Table 1. $\beta_1, \beta_2$ have been calculated for all combinations of $\rho = 8$, 16, 24, 32, $\infty$, and $q_a, q_k = \frac{1}{9}$, $\frac{1}{5}$, $\frac{1}{3}$, $\frac{1}{2}$ and 1. For $\rho = 8, a, k$ take the values of $1, 2, 4, 8, \infty$; for $\rho = 16$ they take the values 2, 4, 8, 16, $\infty$ and so on. The left-hand side of the table gives $\beta_1, \beta_2$ for all combinations of the four finite values of $\rho$ with the values of $q_a, q_k$.

The right-hand side gives $\beta_1, \beta_2$ for the negative binomials of case (ii), in this case taken as $\{1/p_k-(q_k A/p_k)\}^{-a}$ for the same values of $q_k$ and $a$ as on the left-hand side. Here $\beta_1, \beta_2$ are unaltered by the interchange of $a$ and $k$.

On the left-hand side it will be noticed that when $a \neq k$, the same values of $\beta_1, \beta_2$ occur twice (since the interchange of $a$ and $k$ does not alter the distribution). They have deliberately been entered twice. On the right-hand side $\rho \to \infty$ in all cases, so that $\beta_1, \beta_2$ depend only on $a$ and $q_k$. Here the values when $a$ and $k$ are interchanged have not been entered twice, but the values for fixed $q_k$ and $a = 2, 4, 8, 16, \infty$ are of course the same, wherever they occur. The repetition is again deliberate. Its purpose is to enable the reader to have the complete set of values for any row or column; he can then see clearly the effect of varying any one parameter, when the others are kept constant. Comparisons of the right-hand side of the table with its left-hand side show the changes made for fixed $a$ and $q_k$ when $\rho \to \infty$ as compared with the fixed value of $\rho$ in the same row of the table on the left. For instance, when $a = 8$, $q_k = 0{\cdot}3$, the values of $\beta_1, \beta_2$ for $\rho = 16$ are 2·26 and 7·11, but when $\rho \to \infty$ they are 0·67 and 3·92. Apart from the negative binomial, the other limiting cases may also be examined. The case when, for all values of $\rho$, $a = 1$ (or $k = 1$) corresponds to the simple Waring distribution. The continuous analogue of all the distributions on the left is Pearson's Type VI. When $q_a = 1$, $q_k = 1$, for all values of $\rho$, Type V occurs as a special case, which is also a limiting form of the G.W. distribution. The $\beta_1, \beta_2$ of the hypergeometric then satisfy the cubic for Type V. When $0 < q_k \leqslant 1$, $q_a = 1$ and $\rho \to \infty$ the continuous analogue, which here is also the limiting form, is the normal distribution. When $\rho \to \infty$, $a$ is finite and $q_k \to 1$ or vice versa, the continuous analogue and limiting form is Pearson's Type III.

The values in the last column (except for rounding-off errors) satisfy $2\beta_2-3\beta_1-6$ exactly. It should be noted that $2\beta_2-3\beta_1-6>0$ for all the values of $\beta_1, \beta_2$ on the left-hand side of the table, and $<0$ for all the negative binomials. For fixed $a$ and $q_k$ (or $k$ and $q_a$) $2\beta_2-3\beta_1-6$ vanishes for some value of $\rho$, but the continuous analogue remains Type VI, as long as $\rho$ is finite. Other points of interest are mentioned in (i) and (ii) below, where $a$ and $k$ may as usual be interchanged.

(i) For fixed $\rho$ and $a$ (which implies fixed $\rho$ and $q_a$), $\beta_1, \beta_2$ are near their limiting values when $q_k > 0{\cdot}5$.

(ii) For fixed $a$ and $q_k$, the values of $\beta_1, \beta_2$ do not approach their limiting negative binomial values so rapidly. Even when $\rho = 32$, there is a considerable difference; but the change between $q_k = \frac{1}{2}$ and $q_k = 1$ is much less than between $q_k = \frac{1}{9}$ and $q_k = \frac{1}{2}$.

Since the G.W. distribution has its first four moments finite for $\rho > 4$, it seemed reasonable to take $\rho = 8$, as the lowest value of the argument $\rho$ in Table 1. Perhaps the fact that $\rho = 8$ is the largest value of $\rho$ that Karl Pearson would have called heterotypic provides a historical reason for the choice. In any case, Table 1 gives, it seems, an adequate conspectus of $\beta_1, \beta_2$ over the whole range of the parameters $\rho, a, k$, when the first four moments are finite.

It has recently been shown (Irwin, 1968) that the G.W. distribution is of particular interest in the theory of accident distributions, where one would expect the first four moments to be finite, because the negative binomial distribution so often gives a satisfactory fit.

However, the original motive for undertaking this research was to examine the cases where the G.W. distributions have especially long tails. In such cases the mean, or the mean and variance, or the first three or first four moments may be infinite. For this reason the distributions were tabulated on an electronic computer for $\rho = 0{\cdot}5\ (1{\cdot}0)\ 4{\cdot}5$ and appropriate values of $q_a, q_k = 0{\cdot}25, 0{\cdot}5\ (0{\cdot}1)\ 0{\cdot}9$ (see Part II). Of these only $\rho = 4{\cdot}5$ gives a finite fourth moment. The values of $\beta_1, \beta_2$ for $\rho = 4{\cdot}5$ and $q_a, q_k = \frac{1}{9}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1$ are naturally very large. However, as a connecting link between this section and Part II, the values of $\beta_1, \beta_2$ have been calculated for $\rho = 4{\cdot}5$, $q_k = 1$ and $q_a = \frac{1}{9}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}$ and 1. The results are:

| $a$ | $q_a$ | $\beta_1$ | $\beta_2$ |
|---|---|---|---|
| 0·5625 | $0{\cdot}\dot{1}$ | 41·60 | 202·21 |
| 1·125 | 0·2 | 28·24 | 142·09 |
| 2·25 | $0{\cdot}\dot{3}$ | 21·99 | 113·94 |
| 4·5 | 0·5 | 19·29 | 101·81 |
| $\infty$ | 1·0 | 17·78 | 95·00 |

These are the smallest set of values for $\rho = 4{\cdot}5$ and the same values of $q_a$ and $q_k$ as in Table 1.

## 5. The Mode

5.1. The values of the variate are here taken to be $x = 0, 1, 2, \ldots, r$ and the corresponding frequencies are denoted by $f_x$. By examining the ratio $f_x/f_{x-1}$, it is easily shown that the value of $x$ corresponding to the greatest frequency is the *integer*

$$r = I\{(a-1)(k-1)/(\rho+1)\}, \tag{5.1}$$

where $I(z)$ denotes the greatest integer not greater than $z$. If $r = 0$, the curve is J-shaped. We *could* say that $r = 0$ gives the mode, but, as we wish to distinguish distributions with a genuine mode from those which are J-shaped, we here reserve the term "mode" for the case when $r \geqslant 1$. Let us write $\lambda = (a-1)(k-1)/(\rho+1)$; then, if $\lambda = r \geqslant 1$ exactly, $f_r = f_{r-1}$ and both are greater than any other frequency. In this case we still call $x = r$ the mode. Thus the mode is $r$ where $r \leqslant \lambda < (r+1)$.

TABLE 1

*Values of $\beta_1, \beta_2$ and the mode for the Generalized Waring Distribution*

| $\rho$ | $a$ | $q_a$ | | Values for $q_k = 0{\cdot}\dot{1}, 0{\cdot}2, 0{\cdot}\dot{3}, 0{\cdot}5, 1{\cdot}0$ | | | | | Values for negative binomial $\{(1/p_k-(q_k/p_k)\}^{-a}$ $\rho\to\infty, k\to\infty$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $q_k=$ | 0·1̇ | 0·2 | 0·3̇ | 0·5 | 1·0 | 0·1̇ | 0·2 | 0·3̇ | 0·5 | 1·0 |
| | | | $k=$ | 1 | 2 | 4 | 8 | ∞ | | | | | |
| 8 | 1 | 0·1̇ | $\beta_1$ | 24·60 | 16·34 | 12·43 | 10·71 | 9·72 | 11·11 | 7·20 | 5·33 | 4·50 | 4·00 |
| | | | $\beta_2$ | 46·61 | 33·34 | 27·07 | 24·32 | 22·72 | 16·11 | 12·20 | 10·33 | 9·50 | 9·00 |
| | | | Mode | 0 | 0 | 0 | 0 | ∞ | 0 | 0 | 0 | 0 | ∞ |
| | 2 | 0·2 | $\beta_1$ | 16·34 | 10·85 | 8·25 | 7·11 | 6·45 | 5·56 | 3·60 | 2·67 | 2·25 | 2·00 |
| | | | $\beta_2$ | 33·34 | 24·04 | 19·64 | 17·72 | 16·60 | 9·56 | 7·60 | 6·67 | 6·25 | 6·00 |
| | | | Mode | 0 | 0 | 0 | 0 | ∞ | 0 | 0 | 0 | 1 | ∞ |
| | 4 | 0·3̇ | $\beta_1$ | 12·43 | 8·25 | 6·28 | 5·41 | 4·91 | 2·78 | 1·80 | 1·33 | 1·12 | 1·00 |
| | | | $\beta_2$ | 27·07 | 19·64 | 16·13 | 14·60 | 13·70 | 6·28 | 5·30 | 4·83 | 4·62 | 4·50 |
| | | | Mode | 0 | 0 | 1 | 2 | ∞ | 0 | 0 | 2 | 3 | ∞ |
| | 8 | 0·5 | $\beta_1$ | 10·71 | 7·11 | 5·41 | 4·66 | 4·23 | 1·39 | 0·90 | 0·67 | 0·56 | 0·50 |
| | | | $\beta_2$ | 24·32 | 17·72 | 14·60 | 13·23 | 12·44 | 4·64 | 4·15 | 3·92 | 3·81 | 3·75 |
| | | | Mode | 0 | 0 | 2 | 5 | ∞ | 0 | 2 | 5 | 7 | ∞ |
| | ∞ | 1·0 | $\beta_1$ | 9·72 | 6·45 | 4·91 | 4·23 | 3·84 | 0 | 0 | 0 | 0 | 0 |
| | | | $\beta_2$ | 22·72 | 16·60 | 13·70 | 12·44 | 11·70 | 3 | 3 | 3 | 3 | 3 |
| | | | Mode | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ |
| | | | $k=$ | 2 | 4 | 8 | 16 | ∞ | | | | | |
| 16 | 2 | 0·1̇ | $\beta_1$ | 9·34 | 6·12 | 4·59 | 3·92 | 3·52 | 5·56 | 3·60 | 2·67 | 2·25 | 2·00 |
| | | | $\beta_2$ | 16·97 | 12·69 | 10·65 | 9·75 | 9·22 | 9·56 | 7·60 | 6·67 | 6·25 | 6·00 |
| | | | Mode | 0 | 0 | 0 | 0 | ∞ | 0 | 0 | 0 | 1 | ∞ |
| | 4 | 0·2 | $\beta_1$ | 6·12 | 4·01 | 3·01 | 2·57 | 2·31 | 2·78 | 1·80 | 1·33 | 1·12 | 1·00 |
| | | | $\beta_2$ | 12·69 | 9·68 | 8·25 | 7·62 | 7·25 | 6·28 | 5·30 | 4·83 | 4·62 | 4·50 |
| | | | Mode | 0 | 0 | 1 | 2 | ∞ | 0 | 0 | 2 | 3 | ∞ |
| | 8 | 0·3̇ | $\beta_1$ | 4·59 | 3·01 | 2·26 | 1·93 | 1·73 | 1·39 | 0·90 | 0·67 | 0·56 | 0·50 |
| | | | $\beta_2$ | 10·65 | 8·25 | 7·11 | 6·61 | 6·31 | 4·64 | 4·15 | 3·92 | 3·81 | 3·75 |
| | | | Mode | 0 | 1 | 2 | 6 | ∞ | 0 | 2 | 5 | 7 | ∞ |
| | 16 | 0·5 | $\beta_1$ | 3·92 | 2·57 | 1·93 | 1·64 | 1·48 | 0·69 | 0·45 | 0·33 | 0·28 | 0·25 |
| | | | $\beta_2$ | 9·75 | 7·62 | 6·61 | 6·16 | 5·90 | 3·82 | 3·58 | 3·46 | 3·41 | 3·38 |
| | | | Mode | 0 | 2 | 6 | 13 | ∞ | 2 | 4 | 11 | 15 | ∞ |
| | ∞ | 1·0 | $\beta_1$ | 3·52 | 2·31 | 1·73 | 1·48 | 1·33 | 0 | 0 | 0 | 0 | 0 |
| | | | $\beta_2$ | 9·22 | 7·25 | 6·31 | 5·90 | 5·66 | 3 | 3 | 3 | 3 | 3 |
| | | | Mode | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ |

5.2. If $a, k$ are fixed it follows from (5.1) that the smaller $\rho$ is, the bigger is the mode. The same property may also be expressed as follows:

If $x = r$ is the mode and $a, k$ are fixed,

$$\frac{(a-1)(k-1)}{r} \geqslant (\rho+1) > \frac{(a-1)(k-1)}{(r+1)}. \tag{5.2}$$

As an example, Table 2 shows the limits for $\rho$ when $(a-1)(k-1) = 1{\cdot}5$, 2, 2·5, 3, 4, 5, 10 and $r = 1\ (1)\ 9$.

TABLE 1 (*cont.*)

| $\rho$ | $a$ | $q_a$ | | *Values for* $q_k$ = 0·1, 0·2, 0·3, 0·5, 1·0 | | | | | *Values for negative binomial* $\{(1/p)_k-(q_k/p_k)\}^{-a}$ $\rho\to\infty$, $k\to\infty$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $q_k$ = | 0·1̇ | 0·2 | 0·3̇ | 0·5 | 1·0 | 0·1̇ | 0·2 | 0·3̇ | 0·5 | 1·0 |
| | | | $k$ = | 3 | 6 | 12 | 24 | ∞ | | | | | |
| 24 | 3 | 0·1̇ | $\beta_1$ | 5·80 | 3·79 | 2·83 | 2·40 | 2·15 | 3·70 | 2·40 | 1·78 | 1·50 | 1·33 |
| | | | $\beta_2$ | 11·33 | 8·77 | 7·55 | 7·01 | 6·69 | 7·37 | 6·07 | 5·44 | 5·17 | 5·00 |
| | | | Mode | 0 | 0 | 0 | 1 | ∞ | 0 | 0 | 1 | 2 | ∞ |
| | 6 | 0·2 | $\beta_1$ | 3·79 | 2·47 | 1·85 | 1·57 | 1·40 | 1·85 | 1·20 | 0·89 | 0·75 | 0·67 |
| | | | $\beta_2$ | 8·77 | 6·97 | 6·12 | 5·74 | 5·51 | 5·19 | 4·53 | 4·22 | 4·08 | 4·00 |
| | | | Mode | 0 | 0 | 2 | 4 | ∞ | 0 | 1 | 3 | 5 | ∞ |
| | 12 | 0·3̇ | $\beta_1$ | 2·83 | 1·85 | 1·38 | 1·17 | 1·05 | 0·93 | 0·60 | 0·44 | 0·38 | 0·33 |
| | | | $\beta_2$ | 7·55 | 6·12 | 5·44 | 5·13 | 4·95 | 4·09 | 3·77 | 3·61 | 3·54 | 3·50 |
| | | | Mode | 0 | 2 | 4 | 10 | ∞ | 1 | 3 | 8 | 11 | ∞ |
| | 24 | 0·5 | $\beta_1$ | 2·40 | 1·57 | 1·17 | 1·00 | 0·89 | 0·46 | 0·30 | 0·22 | 0·19 | 0·17 |
| | | | $\beta_2$ | 7·01 | 5·74 | 5·13 | 4·86 | 4·70 | 3·55 | 3·38 | 3·31 | 3·27 | 3·25 |
| | | | Mode | 1 | 4 | 10 | 21 | ∞ | 3 | 7 | 17 | 23 | ∞ |
| | ∞ | 1·0 | $\beta_1$ | 2·15 | 1·40 | 1·05 | 0·89 | 0·80 | 0 | 0 | 0 | 0 | 0 |
| | | | $\beta_2$ | 6·69 | 5·51 | 4·95 | 4·70 | 4·56 | 3 | 3 | 3 | 3 | 3 |
| | | | Mode | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ |
| | | | $k$ = | 4 | 8 | 16 | 32 | ∞ | | | | | |
| 32 | 4 | 0·1̇ | $\beta_1$ | 4·21 | 2·74 | 2·05 | 1·73 | 1·55 | 2·78 | 1·80 | 1·33 | 1·12 | 1·00 |
| | | | $\beta_2$ | 8·94 | 7·11 | 6·24 | 5·85 | 5·38 | 6·28 | 5·30 | 4·83 | 4·62 | 4·50 |
| | | | Mode | 0 | 0 | 1 | 2 | ∞ | 0 | 0 | 2 | 3 | ∞ |
| | 8 | 0·2 | $\beta_1$ | 2·74 | 1·79 | 1·33 | 1·13 | 1·01 | 1·39 | 0·90 | 0·67 | 0·56 | 0·50 |
| | | | $\beta_2$ | 7·11 | 5·83 | 5·22 | 4·95 | 4·78 | 4·64 | 4·15 | 3·92 | 3·81 | 3·75 |
| | | | Mode | 0 | 1 | 3 | 6 | ∞ | 0 | 2 | 5 | 7 | ∞ |
| | 16 | 0·3̇ | $\beta_1$ | 2·05 | 1·33 | 0·99 | 0·84 | 0·75 | 0·69 | 0·45 | 0·33 | 0·28 | 0·25 |
| | | | $\beta_2$ | 6·24 | 5·22 | 4·73 | 4·51 | 4·38 | 3·82 | 3·58 | 3·46 | 3·41 | 3·38 |
| | | | Mode | 1 | 3 | 6 | *n*4 | ∞ | 2 | 4 | 11 | 15 | ∞ |
| | 32 | 0·5 | $\beta_1$ | 1·73 | 1·13 | 0·84 | 0·71 | 0·64 | 0·35 | 0·22 | 0·17 | 0·14 | 0·12 |
| | | | $\beta_2$ | 5·85 | 4·95 | 4·51 | 4·32 | 4·21 | 3·41 | 3·29 | 3·23 | 3·20 | 3·19 |
| | | | Mode | 2 | 6 | 14 | 29 | ∞ | 4 | 9 | 17 | 31 | ∞ |
| | ∞ | 1·0 | $\beta_1$ | 1·55 | 1·01 | 0·75 | 0·64 | 0·57 | 0 | 0 | 0 | 0 | 0 |
| | | | $\beta_2$ | 5·38 | 4·78 | 4·38 | 4·21 | 4·10 | 3 | 3 | 3 | 3 | 3 |
| | | | Mode | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ | ∞ |

Table 2 makes two points quite clear. First, if $(a-1)(k-1)$ is fixed, the mode cannot be greater than $I'\{(a-1)(k-1)\}$, where $I'(z)$ denotes the greatest integer less than $z$.

Secondly, for fixed $(a-1)(k-1)$, the mode is a very low value of the variate, compared with the effective range of the distribution. Thus, if the mode is at $(a-1)(k-1)/5$ or some larger value which must $\leqslant I'\{(a-1)(k-1)\}$, $\rho\leqslant 4$. The fourth moment is infinite and the distribution has a very long tail. The example in Table 3 illustrates this.

TABLE 2

*Limits for* $\rho$ *when the mode is given and* $(a-1)(k-1)$ *is fixed* $(L<\rho\leqslant U)$

| *Mode* ($r$) | $(a-1)(k-1)$ | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1·5 | | 2 | | 2·5 | | 3 | | 4 | | 5 | | 10 | |
| | *L* | *U* | *L* | *U* | *L* | *U* | *L* | *U* | *L* | *U* | *L* | *U* | *L* | *U* |
| 1 | 0 | 0·5 | 0 | 1 | $\frac{1}{4}$ | $1\frac{1}{2}$ | $\frac{1}{2}$ | 2 | 1 | 3 | $1\frac{1}{2}$ | 4 | 4 | 9 |
| 2 | — | — | — | — | 0 | $\frac{1}{4}$ | 0 | $\frac{1}{2}$ | $\frac{1}{3}$ | 1 | $\frac{2}{3}$ | $1\frac{1}{2}$ | $2\frac{1}{3}$ | 4 |
| 3 | — | — | — | — | — | — | — | — | 0 | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{2}{3}$ | $1\frac{1}{2}$ | $2\frac{1}{3}$ |
| 4 | — | — | — | — | — | — | — | — | — | — | 0 | $\frac{1}{4}$ | 1 | $1\frac{1}{2}$ |
| 5 | — | — | — | — | — | — | — | — | — | — | — | — | $\frac{2}{3}$ | 1 |
| 6 | — | — | — | — | — | — | — | — | — | — | — | — | $\frac{3}{7}$ | $\frac{2}{3}$ |
| 7 | — | — | — | — | — | — | — | — | — | — | — | — | $\frac{1}{4}$ | $\frac{3}{7}$ |
| 8 | — | — | — | — | — | — | — | — | — | — | — | — | $\frac{1}{9}$ | $\frac{1}{4}$ |
| 9 | — | — | — | — | — | — | — | — | — | — | — | — | 0 | $\frac{1}{9}$ |

TABLE 3

*Certain values of mode, mean and S.D., for fixed* $(a-1)(k-1)$

| $(a-1)(k-1)$ | $\rho$ | $a$ | $k$ | *Mode* | *Mean* | *S.D.* |
|---|---|---|---|---|---|---|
| 10 | 4 | 6 | 3 | 2 | 6 | 7·3 |
| 10 | 2·5 | 6 | 3 | 2 | 12 | 23·2 |
| 10 | 2 | 6 | 3 | 3 | 18 | ∞ |
| 100 | 4 | 21 | 6 | 20 | 42 | 38·9 |
| 100 | 2·5 | 21 | 6 | 28 | 84 | 137·5 |
| 100 | 2 | 21 | 6 | 33 | 126 | ∞ |

Distributions which are "heterotypic" in Karl Pearson's sense have $\rho\leqslant 8$. For these the mode cannot be less than $I\{(a-1)(k-1)/9\}$ or greater than $I'\{(a-1)(k-1)\}$. If the mode has its greatest possible value, $\rho\leqslant 1/\{(a-1)(k-1)-1\}$; for example, if $(a-1)(k-1)=100$, $\rho\leqslant\frac{1}{99}$ and the tail is very long indeed.

5.3. The position is quite different if $q_a, q_k$ are constant. Since

$$a = \rho q_a/p_a = \rho/u, \quad \text{say,}$$
$$k = \rho q_k/p_k = \rho/v, \quad \text{say,}$$

we have

$$\lambda = (a-1)(k-1)/(\rho+1) = \{(\rho/u)-1\}\{(\rho/v)-1\}/(\rho+1) \tag{5.3}$$

or

$$(\rho-u)(\rho-v) = (\rho+1)\,\lambda uv, \quad \text{where } r\leqslant\lambda<(r+1), \tag{5.4}$$

which gives

$$\rho = \tfrac{1}{2}\{u+v+uv\lambda\}\left\{1+\sqrt{\left(1+\frac{4uv(\lambda-1)}{(u+v+uv\lambda)^2}\right)}\right\}. \tag{5.5}$$

Since $\rho>0$ we must take the positive sign for the square root in (5.5). Here, for fixed $u, v$, the mode increases with $\rho$.

2

If $\lambda = 0$, we know the distribution is J-shaped; if $\lambda > 0$ there is a mode. It follows that $x = r$ is the mode if

$$\tfrac{1}{2}\{u+v+uvr\}\left\{1+\sqrt{\left(1+\frac{4uv(r-1)}{(u+v+uvr)^2}\right)}\right\}$$
$$\leqslant \rho$$
$$< \tfrac{1}{2}\{u+v+uv(r+1)\}\left\{1+\sqrt{\left(1+\frac{4uvr}{(u+v+uv(r+1))^2}\right)}\right\}. \quad (5.6)$$

Hence the range within which $\rho$ lies has a length roughly equal to $uv$ and increases with $uv$; that is to say it increases as $q_a q_k/p_a p_k$ diminishes.

For example, when $\rho = 8$, $q_a = \frac{1}{3}$, $q_k = \frac{1}{2}$, $a = 4$, $k = 8$; an application of equation (5.1) gives $x = 2$ for the mode. Now suppose, keeping $q_a = \frac{1}{3}$, $q_k = \frac{1}{2}$, that the mode is at $x = 2$. Equation (5.6), putting $u = 2$, $v = 1$ gives $7{\cdot}275 \leqslant \rho < 9{\cdot}424$. The range is $2{\cdot}149$, while $uv = 2$. When $\rho = 8$, $u = \frac{1}{9}$, $v = \frac{1}{9}$, i.e. $q_a = q_k = \frac{9}{10}$, $a = k = 72$, we find from equation (5.1) that the mode is at $x = 560$. If the mode is 560 and $q_a = q_k = \frac{9}{10}$, equation (5.6) gives $7{\cdot}9986 \leqslant \rho < 8{\cdot}0111$. The range is $0{\cdot}0125$ while $uv = \frac{1}{81} = 0{\cdot}0123$.

5.4. Table 1, Section 4, primarily a table of $\beta_1, \beta_2$, also gives the values of the mode. Here the entry zero means that the distribution is J-shaped. If $a = 1$ or $k = 1$ (the simple Waring case) the distributions are always J-shaped.

The table shows that if $a = 1$ or $k = 1$ (the simple Waring case) or indeed if either $a$ or $k$ have any constant values, there is not much difference between the positions of the mode for distributions with a fixed $\rho$ and for the corresponding negative binomials ($\rho \to \infty$, $k \to \infty$, $q_k$ fixed). The same is true if $a$ and $k$ are interchanged. If $a > 1$, $k > 1$, the following results may be noted:

(i) If $q_a, q_k$ are both fixed there is a mode for sufficiently large $\rho$. Table 1 shows how the mode increases with $\rho$. For values up to $q_a = 0{\cdot}5$, $q_k = 0{\cdot}5$, we can see from the table that this is at a roughly constant rate. That the same is true for all $q_a, q_k$ can be seen from equation (5.6) on p. 19; the rate of increase of the mode with $\rho$ is roughly equal to $1/uv$ or $q_a q_k/p_a p_k$.

(ii) If $\rho$ and $q_k$ are fixed, there is a mode for sufficiently large $q_a$. Table 1 shows how the mode increases with $q_a$ for fixed $q_k \leqslant 0{\cdot}5$.

Table 1 deals with relatively large values of $\rho$, distributions in which at least the first seven moments are finite. Since this study was mainly directed to long-tailed distributions, which also have a mode, we have to consider the mode of distributions calculated on the electronic computer. These values of the mode are given in Part II and will be considered in relation to the values of the median, mean and the characteristics of the distributions. Here it is only necessary to say that the table illustrates the facts that

(i) If $a = \rho q_a/p_a \leqslant 1$ or if $k = \rho q_k/p_k \leqslant 1$, the distributions are certainly J-shaped. If $a > 1$ and $k > 1$ the distributions are J-shaped if $\rho > (a-1)(k-1)-1$ or, equivalently, $\rho < \{1/(q_a q_k) - 1\}$. Otherwise there is a mode.

(ii) Where there is a mode, the rule, that the mode increases by approximately $q_a q_k/p_a p_k$ per unit increase in $\rho$. rapidly becomes true as $\rho$ increases; even for such small values of $\rho$ as those here tabulated. For example, if $q_a = 0{\cdot}5$, $q_k = 0{\cdot}9$, $q_a q_k/p_a p_k = 9$; the actual increases are 7 for $\rho = 1{\cdot}5 - 2{\cdot}5$, 8 for $\rho = 2{\cdot}5 - 3{\cdot}5$ and

3·5−4·5. If $q_a = 0{\cdot}8$, $q_k = 0{\cdot}9$, $q_a q_k/p_a p_k = 36$. The actual increases are 23 for $\rho = 0{\cdot}5 - 1{\cdot}5$, 30 for $\rho = 1{\cdot}5 - 2{\cdot}5$, 33 for $\rho = 2{\cdot}5 - 3{\cdot}5$, 34 for $\rho = 3{\cdot}5 - 4{\cdot}5$.

[To be continued]

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† In the references to Karl Pearson's papers in *Phil. Trans.* it was not possible to obtain the exact pages to which reference is made in this paper. These papers were, however, reprinted in book form in 1956. The exact pages in the book are indicated.