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The Generalized Waring Distribution. Part III

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(*Retired*)

SUMMARY

Here the continuous analogue of the Generalized Waring Distribution is obtained. It is in general of Pearson's Type VI and the successive moments of the G.W. distribution and the corresponding Type VI become infinite for the same values of ρ. When both $a \to \infty$ and $k \to \infty$ the continuous analogue takes the Type V form. Both in this case and the case when only one of $a, k \to \infty$ the continuous analogue can be regarded as a *limiting form* of the distribution. When $q_a \to 1$, $\rho \to \infty$, k remaining finite (a and k may be interchanged), the continuous analogue or *limiting form* becomes Type III.

Exceptions to the Type VI form occur in certain cases, when the point (a, k) is in the square $0 < a \leqslant 1$, $0 < k \leqslant 1$. If the point (a, k) is in the area common to the square and the circle $\{a+\frac{1}{2}(\rho-1)\}^2+\{k+\frac{1}{2}(\rho-1)\}^2 = \frac{1}{4}(\rho+1)^2$ the continuous analogue is Pearson's Type IV (Type V on the circle). The system of circles when ρ varies is discussed; a diagram shows the sub-division of the square by the circles. According to the values of ρ, the form may be Type VI in one continuous region, or three or four separate regions, of the square.

A short appendix deals with the case when a, k are complex conjugate, $\gamma \pm i\delta$. The G.W. distribution remains real. The discriminating circles are now replaced by rectangular hyperbolae in two real variables $a' = \gamma+\delta$, $k' = \gamma-\delta$. The hyperbolae have their centres at $a' = -\frac{1}{2}(\rho-1)$, $k' = -\frac{1}{2}(\rho-1)$. The continuous analogue will be of Type VI or Type IV according as the point (a', k') lies on the opposite or the same side of the hyperbola as the centre. If the point lies on the hyperbola, it will be Type V.

The analysis used here suggests the possibility, if a, k are real but one or both may be *negative*, of classifying all Pearson's Types according to the values of the parameters of the hypergeometric from which they are derived.

Keywords: CONTINUOUS ANALOGUES; EXCEPTIONS TO TYPE VI FORM; CRITERIA FOR TYPE; DIAGRAMMATIC REPRESENTATION; COMPLEX CONJUGATE PARAMETERS

Relation of the Generalized Waring Distribution to Karl Pearson's System of Frequency Distributions

8. The Slope/Ordinate Method†

8.1. Karl Pearson (1895) originally determined his system of frequency curves by equating $(1/y)(dy/dx)$ to the ratio of slope to ordinate in the histogram of the hypergeometric distribution and integrating the resultant differential equation.

Later, however, he expressed the constants in his differential equation in terms of $\sigma^2 = \mu_2$, $\beta_1 = \mu_3^2/\mu_2^3$, $\beta_2 = \mu_4/\mu_2^2$ where $\beta_1, \beta_2, \mu_2, \mu_3, \mu_4$ refer to the theoretical curve. He then fitted, by equating the theoretical mean μ' the standard deviation $\sigma = \sqrt{\mu_2}$, β_1 and β_2 to the corresponding functions obtained from the data. This was natural at the time, for his object was to graduate the frequency distributions of observations.

† The numbering of Sections continues from Part II.

He certainly knew, though I do not think he ever explicitly stated in print, that the continuous curve obtained by the slope-ordinate method had not the same constants as that obtained by equating the β_1, β_2 of the curve and hypergeometric.

Certain distributions in his system had infinite moments. If the eighth and higher moments were infinite he called the curve heterotypic, because then the fourth moment of the data would have an infinite standard error and be useless for fitting.

It is plausible to suppose that he had not realized the possibility that a theoretical frequency distribution with, say, infinite mean and variance, *could* provide a satisfactory description of some observed phenomenon; this was very likely due to the method of fitting he adopted. For he would have argued that the moments of any empirical distribution are necessarily finite, and took it for granted that these could be used as satisfactory estimates of the theoretical moments, not realizing, perhaps, that if samples were taken from a universe with, say, μ_4 infinite, the fourth moment of the data would, while finite, increase with the size of sample, tending to infinity with sample size.

If his object had been to approximate to a discrete or grouped continuous distribution by a theoretical curve, and if he had kept to the slope ordinate method, he would have had no difficulty with infinite moments.

Here we are particularly concerned with the long-tailed G.W. distributions which may have even the mean or the variance and all higher moments infinite. We therefore define the curves obtained by the slope-ordinate method to be the *continuous analogue* of the discrete G.W. distribution. In certain circumstances to be described below, the continuous analogue can be regarded as a *limiting form* of the distribution. Also, as will be shown, the continuous analogue is, in general, Pearson's Type VI which may be written

$$y = Cx^{q_2}(x+a)^{-q_1}, \quad 0 \leqslant x < \infty, \quad q_1 > 0, \quad q_2 > -1.$$

If the slope-ordinate method is used, we find $\rho = q_2 - q_1 \dotplus 1$ and the moments of the continuous analogue become infinite at the same value of ρ as the G.W. distribution itself. This does not hold good if the Pearsonian curve is obtained by equating the β_1, β_2 to those of the G.W. distribution.

8 (CONT.). THE CONTINUOUS ANALOGUE

8.2. We now derive the *continuous analogue* by the slope-ordinate ratio method. If the frequencies are f_r $(r = 0, 1, 2, 3, \ldots)$,

$$f_r = \frac{Ca_{[r]}\,k_{[r]}}{(a+k)_{[r]}}, \tag{8.1}$$

$$\frac{f_r}{f_{r-1}} = \frac{(a+r-1)(k+r-1)}{r(\rho+a+k+r-1)}, \tag{8.2}$$

$$\frac{f_r - f_{r-1}}{f_r} = \frac{-r(\rho+1)+(a-1)(k-1)}{r^2+r(\rho+a+k-1)}, \tag{8.3}$$

while

$$\frac{\frac{1}{2}(f_r+f_{r-1})}{f_r} = \frac{r^2+r(\frac{1}{2}\rho+a+k-\frac{3}{2})+\frac{1}{2}(a-1)(k-1)}{r^2+r(\rho+a+k-1)}. \tag{8.4}$$

Thus

$$\frac{f_r - f_{r-1}}{\frac{1}{2}(f_r + f_{r-1})} = \frac{(a-1)(k-1)-(\rho+1)r}{r^2+\frac{1}{2}r(\rho+2(a+k)-3)+\frac{1}{2}(a-1)(k-1)} \tag{8.5}$$

and this is the slope-ordinate ratio at $x = r-\frac{1}{2}$. This is equated to

$$\left(\frac{1}{y}\frac{dy}{dx}\right)_{x=r-\frac{1}{2}} = \frac{\alpha\kappa-(\rho+1)r}{r^2+\frac{1}{2}r\{\rho+2(\alpha+\kappa)+1\}+\frac{1}{2}\alpha\kappa}, \tag{8.6}$$

where $\alpha = a-1$, $\kappa = k-1$.

Putting $x = X-\frac{1}{2}$, we find

$$\left(\frac{1}{y}\frac{dy}{dX}\right)_{X=r} = \frac{\alpha\kappa-(\rho+1)X}{X^2+\frac{1}{2}\{\rho+2(\alpha+\kappa)+1\}X+\frac{1}{2}\alpha\kappa}. \tag{8.7}$$

If we now write

$$\frac{1}{y}\frac{dy}{dX} = \frac{\alpha\kappa-(\rho+1)X}{(X+a_1)(X+a_2)} = \frac{\alpha\kappa+(\rho+1)a_2}{(a_1-a_2)(X+a_2)} - \frac{\alpha\kappa+(\rho+1)a_1}{(a_1-a_2)(X+a_1)} \tag{8.8}$$

so that $-a_1$ and $-a_2$ $(a_1 > 0, a_1 > a_2)$ are the roots of the denominator of (8.7), we find

$$\log y = \log C + \frac{\alpha\kappa+(\rho+1)a_2}{(a_1-a_2)}\log(X+a_2) - \frac{\alpha\kappa+(\rho+1)a_1}{(a_1-a_2)}\log(X+a_1) \tag{8.9}$$

and

$$y = C(X+a_2)^{q_2}(X+a_1)^{-q_1} \tag{8.10}$$

or, writing $X = -a_2+\xi$

$$y = C\xi^{q_2}(\xi+A)^{-q_1}, \tag{8.11}$$

where $A = a_1-a_2$, and it is easily shown that

$$C = \Gamma(q_1)/A^{q_2-q_1+1}\,\Gamma(q_2+1)(q_1-q_2-1).$$

This is Pearson's Type VI, provided a_1 and a_2 are real, as will be shown usually to be true.

In (8.11)

$$q_2 = \frac{\alpha\kappa+(\rho+1)a_2}{(a_1-a_2)}, \quad q_1 = \frac{\alpha\kappa+(\rho+1)a_1}{(a_1-a_2)}. \tag{8.12}$$

This shows that if a_1 and a_2 are real $a_1 > a_2$, then $q_1 > q_2$. Now, from (8.7), $a_1+a_2 = \frac{1}{2}\{\rho+2(\alpha+\kappa)+1\} = \frac{1}{2}R$ (say), $a_1 a_2 = \frac{1}{2}(\alpha\kappa)$; hence

$$\left.\begin{aligned} a_1 &= \tfrac{1}{4}\{R+\surd(R^2-8\alpha\kappa)\}, \\ a_2 &= \tfrac{1}{4}\{R-\surd(R^2-8\alpha\kappa)\}, \end{aligned}\right\} \tag{8.13}$$

whence, from (8.13), it follows that

$$\left.\begin{aligned} q_1 &= \tfrac{1}{2}(\rho+1)+\frac{\frac{1}{2}\{(\rho+1)R+4\alpha\kappa\}}{\surd(R^2-8\alpha\kappa)}, \\ -q_2 &= \tfrac{1}{2}(\rho+1)-\frac{\frac{1}{2}\{(\rho+1)R+4\alpha\kappa\}}{\surd(R^2-8\alpha\kappa)}. \end{aligned}\right\} \tag{8.14}$$

Thus, a_1 and a_2 are real and unequal if $R^2 > 8\alpha\kappa$; real and equal if $R^2 = 8\alpha\kappa$ and complex conjugate if $R^2 < 8\alpha\kappa$. If $a > 1$, $k > 1$, a_1, a_2 are both positive and $R^2 > 8\alpha\kappa$ is always true and the Pearson curve is Type VI. The other two cases, which lead to Types V and IV, are exceptional. The criteria for these are discussed in detail below.† Furthermore, the rth moment about the origin of (8.11) is given by

$$\mu_r' = \frac{A^r\,\Gamma(q_1)}{\Gamma(q_2+1)\,\Gamma(q_1-q_2-1)}\int_0^\infty x^{q_2+r}(1+x)^{-q_1}\,dx. \tag{8.15}$$

Writing $x = (1/u) - 1$, we find

$$\mu_r' = \frac{A^r\,\Gamma(q_1)}{\Gamma(q_2+1)\,\Gamma(q_1-q_2-1)}\int_0^1 (1-u)^{q_2+r}\,u^{q_1-q_2-r-2}\,du, \tag{8.16}$$

$$= \frac{A^r\,\Gamma(q_1)}{\Gamma(q_2-1)\,\Gamma(\rho)}\int_0^1 (1-u)^{q_2+r}\,u^{\rho-r-1}\,du$$

$$= \frac{A^r\,\Gamma(q_2+r+1)\,\Gamma(\rho-r)}{\Gamma(q_2+1)\,\Gamma(\rho)}$$

$$= \frac{A^r(q_2+1)(q_2+2)\ldots(q_2+r)}{(\rho-1)(\rho-2)\ldots(\rho-r)}. \tag{8.17}$$

Thus the successive moments of the Type VI curve (8.10 or 8.11) are infinite when $\rho \leqslant r$, $r = 1, 2, 3, \ldots, \infty$. This shows that the moments of the continuous analogue become infinite for the same values of ρ as the corresponding discrete G.W. distribution.

In the particular case when $k \to \infty$ we find $q_1 = (\rho+a)$, $q_2 = (a-1)$. Writing ξ in place of ξ/A, (8.11) can be written

$$y\,d\xi = \frac{\Gamma(\rho+a)}{\Gamma(a)\,\Gamma(\rho)}\,\xi^{a-1}(1+\xi)^{-(\rho+a)}\,d\xi \tag{8.18}$$

with $\mu_2 = \Sigma^2 = a(\rho+a+1)/(\rho-1)^2(\rho-2)$ in agreement with equation (4.5) of Part I.

If now $a \to \infty$ and we write $\xi = aX$, (8.18) assumes the Type V form

$$y\,dX = \frac{1}{\Gamma(\rho)}\,e^{-1|X}\,X^{-(\rho+1)}\,dX. \tag{8.19}$$

We have now established the results stated earlier (Part I, p. 23) that the continuous analogue of the G.W. distribution is in general Type VI, and that the successive moments of the G.W. distribution and the corresponding Type VI become infinite for the same values of ρ; that it assumes the form (8.18) when $k \to \infty$ and that when

† It is easy to show algebraically that if $R^2 > 8\alpha\kappa$ then (i) $q_1 > 0$, (ii) $q_2 > -1$. If (i) is true (ii) follows immediately from $q_1 - q_2 - 1 = \rho > 0$. If $\alpha\kappa$ is negative, then only one of α, κ can be negative. Say this is $\alpha = -\alpha'$, then $0 < \alpha' < 1$. Writing $\lambda = \alpha'\kappa \geqslant 0$ it follows that $q_1 > 0$ if $(\rho+1)\sqrt{(R^2+8\lambda)} > 4\lambda - (\rho+1)R$. This is obviously true if $(\rho+1)R \geqslant 4\lambda$. If $(\rho+1)R < 4\lambda$ the condition reduces to $\lambda^2 - \lambda(\rho+1)(\rho+\alpha+\kappa+1) > 0$, i.e. to $(\rho+1)^2 + (\kappa-\alpha')(\rho+1) - \alpha^1\kappa > 0$, that is to $(\rho+1+\kappa)(\rho+1-\alpha^1) > 0$ or $(\rho+a)(\rho+k) > 0$ and this is true since $a \geqslant 0, k \geqslant 0, \rho > 0$. Thus, we always have $q_1 > 0$ and $q_2 > -1$ and Pearson's Type I which would occur for q_1 negative and a_1, a_2 of opposite sign is excluded.

also $a \to \infty$ it assumes the Type V form. Further, since these forms are reached by a change of scale which makes the interval between successive values of the variate tend to zero, they can be regarded as limiting values of the distribution.

The special case when the negative binomial is a limiting form of the G.W. distribution has already been mentioned in Part I, Section 2 (*J. R. Statist. Soc.* A, **138**, Part 1, p. 21). Pearson's Type III is a continuous analogue of the negative binomial only when $q_a \to 1$, $\rho \to \infty$, k remaining finite (where a and k can be interchanged). In general the continuous analogue of the negative binomial is Type VI. In this particular case, Type III is a limiting form of Type VI, and assuming change of scale allowable, a limiting case of the G.W. distribution itself. This can be seen as follows:

If in (8.18) we write $\xi = X/c\rho$ and let $\rho \to \infty$, c being finite, $y d\xi$ in (8.17) becomes

$$\frac{1}{\Gamma(a)}\left(\frac{X}{c}\right)^{a-1} e^{-X/c}\frac{dX}{c}. \tag{8.20}$$

So, when $\rho \to \infty$, a remaining finite, (8.18) assumes the Type III form. This is the limiting form of the negative binomial $\{(1-q_k A^{\lambda})/p_k\}^{-a}$ when $\rho \to \infty$, $k \to \infty$, $q_k \to 1$.

The variance μ_2 becomes $c^2\rho^2(a/\rho^2) = c^2 a$, which agrees with the limiting variance of the negative binomial when $q_k \to 1$ and the variate values are $0, \lambda, 2\lambda, 3\lambda, \ldots$, etc., where $\lambda \to 0$ and $\lambda^2 q_k/p_k^2 = c^2$ is finite.

8 (CONT.). DETAILED CONSIDERATION OF CRITERIA FOR TYPE

8.3. The roots of the denominator on the right-hand side of (8.7) are real if

$$(\rho+1)^2+4(\alpha+\kappa)(\rho+1)+4(\alpha^2+\kappa^2)\geqslant 0 \tag{8.21}$$

or

$$\{\alpha+\tfrac{1}{2}(\rho+1)\}^2+\{\kappa+\tfrac{1}{2}(\rho+1)\}^2-\tfrac{1}{4}(\rho+1)^2\geqslant 0$$

or (8.22)

$$\{a+\tfrac{1}{2}(\rho-1)\}^2+\{k+\tfrac{1}{2}(\rho-1)\}^2\geqslant \tfrac{1}{4}(\rho+1)^2.$$

Thus the roots are real if the point (a,k) is on or outside the circle with centre $\{-\frac{1}{2}(\rho-1), -\frac{1}{2}(\rho-1)\}$ and radius $\frac{1}{2}(\rho+1)$. In this case, the frequency curve is the Pearson Type VI or in the special case when a and k are on the circle, Type V.

The roots are imaginary if (a,k) is within the region common to the interior of the circle (8.22) and the square through the four points (a_i, k_j), $i = 0,1; j = 0,1$. In this case the curve is Pearson's Type IV. The roots are always real in the positive quadrant outside the square.

The circles (8.22) for varying ρ are a family of circles with centre on the diagonal through the origin and whose envelope is the pair of lines $a = 1$, $k = 1$. Let the sides of the square through the origin be denoted by OA, OB; let C be the point of the square diagonally opposite O. Then all the circles touch BC and AC (see Fig. 1).

When $\rho = 0$, the circle (8.22) has its centre at the centre of the square and touches all four sides. When $\rho = 3-2\sqrt{2} = 0{\cdot}172$, the circle has its centre at the point $(\sqrt{2}-1, \sqrt{2}-1)$ or $(0{\cdot}414, 0{\cdot}414)$ and has radius $2-\sqrt{2} = 0{\cdot}586$ and passes through the origin O. It cuts OA again at the point $(2(\sqrt{2}-1), 0)$, i.e. $(0{\cdot}828, 0)$; similarly it cuts OB again at the point $(0, 2(\sqrt{2}-1))$.

When $\rho = 1$, the circle has its centre at the origin and passes through the points A, B, thus dividing the square into two portions. On the lower portion the Pearson curve will be Type IV; elsewhere it will be Type VI (Type V, on the circle).

When $5{\cdot}828 = 3+2\sqrt{2} > \rho > 1$ the circles divide the square into two portions; their centres are in the third quadrant and they touch AC and BC externally to the square. When $\rho = 3+2\sqrt{2}$ the circle passes through the origin O, and no point of it is inside the square. Consequently for $\rho > (3+2\sqrt{2})$ the Pearson curve is always

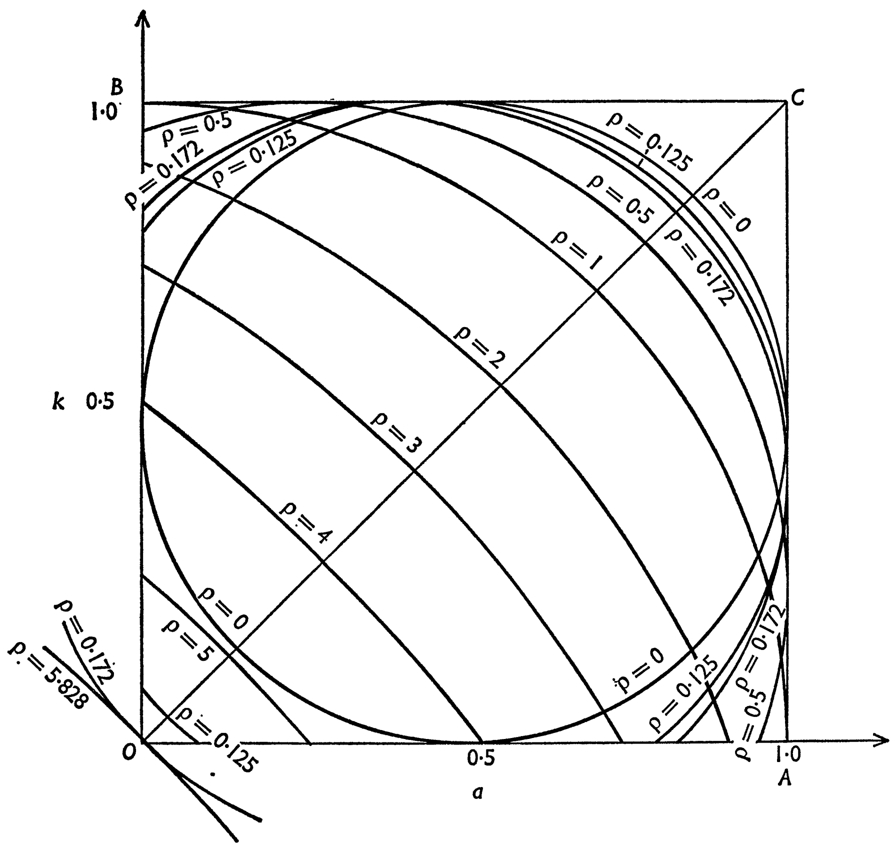

FIG. 1. Circles of the system $\{a+\frac{1}{2}(\rho-1)\}^2+\{k+\frac{1}{2}(\rho-1)\}^2 = \frac{1}{4}(\rho+1)^2$. For given ρ, the continuous analogue of the G.W.D. is Pearson's Type IV if the point a, k is in the area common to the unit square in the first quadrant and the interior of the corresponding circle. On the circle it is Type V. In the rest of the positive quadrant it is Type VI.

Type VI. For $3-2\sqrt{2} < \rho < 1$ the circles touch AC and BC externally and cut OA and OB only once. For instance, when $\rho = \frac{1}{2}$ the circle has its centre at $(\frac{1}{4}, \frac{1}{4})$ and cuts the line OB $(a = 0)$ at $k = 0{\cdot}957$. For $0 < \rho < 3-2\sqrt{2}$ $(= 0{\cdot}172)$ the circle touches AC and BC internally and cuts OA and OB internally twice. For instance, when $\rho = \frac{1}{8}$ the circle has its centre at $(\frac{7}{16}, \frac{7}{16})$ and cuts the line OB $(a = 0)$ at $k = 0{\cdot}109$ and $0{\cdot}816$.

Thus, according to the value of ρ, Type VI may hold in the whole square ($\rho \geqslant 5{\cdot}828$), in one continuous region of the square ($5{\cdot}828 > \rho \geqslant 1$), three separate regions ($1 > \rho \geqslant 0{\cdot}172$) or four separate regions of the square ($0{\cdot}172 > \rho > 0$). On the curved boundaries of the regions the Pearson curve will be Type V† (see Fig. 1).

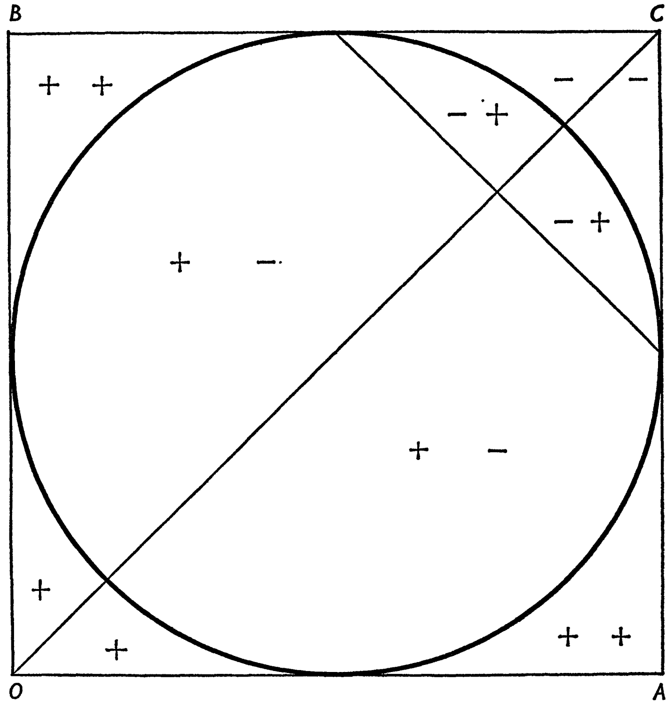

FIG. 2. *OABC* is the unit square in the positive a, k quadrant. Two circles of the system $\{a+\frac{1}{2}(\rho-1)\}^2+\{k+\frac{1}{2}(\rho-1)\}^2 = \frac{1}{4}(\rho+1)^2$ pass through every point inside the square. The two corresponding values of ρ are both positive, both negative or one positive and one negative in the regions shown. Where the signs alternate the sign shown first corresponds to the larger value of $|\rho|$. Along *AC*, *BC*, the two circles are coincident.

8.4. When the continuous analogue is of Type IV, we may write, in (8.9), $a_1 = b+ic$, $a_2 = b-ic$ or

$$\left.\begin{aligned} a_1 &= \tfrac{1}{4}R+\tfrac{1}{4}i\sqrt{(8\alpha\kappa-R^2)}, \\ a_2 &= \tfrac{1}{4}R-\tfrac{1}{4}i\sqrt{(8\alpha\kappa-R^2)}, \end{aligned}\right\} \tag{8.23}$$

† It is interesting to note that two circles of the system $\{a+\frac{1}{2}(\rho-1)\}^2+\{k+\frac{1}{2}(\rho-1)\}^2 = \frac{1}{4}(\rho+1)^2$ pass through every point. They correspond to one positive and one negative value of ρ if the point (a, k) is within the circles $\rho = 0$, i.e. $(a-\frac{1}{2})^2+(k-\frac{1}{2})^2 = \frac{1}{4}$; to two negative values if the point (a, k) is outside the circle and above the line $(a+k) = \frac{3}{2}$; and to two positive values of ρ if (a, k) is outside the circle and below the line $(a+k) = \frac{3}{2}$ (Fig. 2).

Along the lines *AC*, *BC* ($a = 1$ and $k = 1$) the roots of (8.22) are equal and there is only one circle of the system corresponding to points on these lines. There are two circles corresponding to points on *OA*, *OB*.

These results follow because the sum of the roots of (8.22) is $-4(a+k-\frac{3}{2})$ and the product is $4\{(a-\frac{1}{2})^2+(k-\frac{1}{2})^2-\frac{1}{4}\}$. Negative values of ρ are of no particular interest to our problem, for the hypergeometric series does not then converge.

whence we find from (8.7) after a little algebra that

$$\frac{1}{y}\frac{dy}{dX} = \frac{-(\rho+1)(X+\frac{1}{4}R)+\alpha\kappa+\frac{1}{4}(\rho+1)R}{(X+\frac{1}{4}R)^2+\frac{1}{16}(8\alpha\kappa-R^2)} \tag{8.24}$$

and, on integrating,

$$y = \frac{y_0}{\{(X+A)^2+B^2\}^{\frac{1}{2}(\rho+1)}}\exp\left(-\nu\tan^{-1}\frac{X+A}{B}\right), \tag{8.25}$$

where

$$A = \tfrac{1}{4}R, \quad B = \tfrac{1}{4}\sqrt{(8\alpha\kappa-R^2)}, \quad \nu = \frac{-\{4\alpha\kappa+(\rho+1)R\}}{\sqrt{(8\alpha\kappa-R^2)}}. \tag{8.26}$$

We may also write

$$\left.\begin{aligned} A &= \tfrac{1}{4}\{\rho+2(a+k)-3\},\\ B &= \tfrac{1}{4}\sqrt{\{(\rho+1)^2-(2a+\rho-1)^2-(2k+\rho-1)^2\}},\\ -\nu &= \{(\rho+1)^2+2(a+k-2)(a-1)+4(a-1)(k-1)\}/4B. \end{aligned}\right\} \tag{8.27}$$

The curve starts at $X=0$, i.e. $x=-\frac{1}{2}$ and goes to $X=\infty$; in practice the value of y for negative X will usually be negligible, so that we can determine y_0 from the complete integral. We find

$$y_0 = \frac{1}{BF(r,\nu)} \quad \text{where} \quad r=(\rho-1), \quad F(r,\nu) = e^{-\frac{1}{2}\nu\pi}\int_0^\pi \sin^r\phi\, e^{\nu\phi}\,d\phi. \tag{8.28}$$

Tables of $F(r,\nu)$ are given in Table LIV, *Tables for Statisticians and Biometricians* (1924). When the continuous analogue is of Type V, $R^2 = 8\alpha\kappa$, (8.24) becomes

$$\frac{1}{y}\frac{dy}{dX} = \frac{-(\rho+1)(X+\frac{1}{4}R)+\frac{1}{8}\{R^2+2(\rho+1)R\}}{(X+\frac{1}{4}R)^2},$$

whence

$$y = y_0(X+\tfrac{1}{4}R)^{-(\rho+1)}\exp\{-c|(X+\tfrac{1}{4}R)\}, \tag{8.29}$$

where

$$c = \tfrac{1}{8}R\{R+2(\rho+1)\} = \tfrac{1}{8}\{\rho+2(a+k)-3\}\{3\rho+2(a+k)-1\}. \tag{8.30}$$

The actual range is from $X=0$ to $X=\infty$, the complete range is from $X=-\frac{1}{4}R$ to ∞. If the integral between $-\frac{1}{4}R$ and 0 is negligible, then the complete curve may be used and $y_0 = c^\rho/\Gamma(\rho)$, otherwise y_0 can be obtained from a table of the incomplete gamma function.

In fact it can be shown that on the circles within the square of Fig. 1, $|\frac{1}{4}R|\leqslant\sqrt{2}/2 = 0{\cdot}707$, R can be positive or negative, so the start of the curve can be anywhere between $X=0$ and $X=\pm\frac{1}{2}\sqrt{2}$, or between $x=-1{\cdot}207$ and $+0{\cdot}207$ on the original scale of frequency; whereas the G.W.D. plotted as a histogram starts at $X=0$ or $x=-0{,}5$. This suggests that it would be accurate enough, for most purposes, to use $y_0 = c^\rho/\Gamma(\rho)$.

[It is possible for $\frac{1}{4}R$ to be negative. Since $R = \rho+2(a+k)-3$ and $a\geqslant 0$, $k\geqslant 0$, satisfying

$$\{a+\tfrac{1}{2}(\rho-1)\}^2+\{k+\tfrac{1}{2}(\rho-1)\}^2 = \{\tfrac{1}{2}(\rho+1)\}^2 \tag{8.31}$$

and $0 < \rho \leqslant 3+2\surd 2$, it is easy to show, under these conditions, that for $0 \leqslant \rho \leqslant 3-2\surd 2 = 0{\cdot}1712$

$$\min(a+k) = 1-\tfrac{1}{2}\surd 2-(1+\tfrac{1}{2}\surd 2)\rho. \tag{8.32}$$

It follows that

$$\min(\tfrac{1}{4}R) = -\tfrac{1}{4}(1+\surd 2)(\rho+1). \tag{8.33}$$

When $\rho = 0$, this is $-0{\cdot}604$; when $\rho = 3-2\surd 2$ it is $-\tfrac{1}{2}\surd 2 = -0{\cdot}707$. The origin of the Type V curve will then be between $X = 0{\cdot}604$ and $X = 0{\cdot}707$ or $x = 0{\cdot}104$ and $x = 0{\cdot}207$, on the original scale of frequency, whereas the G.W. distribution plotted as a histogram starts at $X = 0$ or $x = -\tfrac{1}{2}$.

However, when $3-2\surd 2 < \rho \leqslant 3+2\surd 2$, $\min(a+k)$ is given by $a = 0$ or $k = 0$ in (8.28) which gives

$$\min(a+k) = \tfrac{1}{2}(1-\rho)+\surd\rho,$$

where the positive square root is taken, that is

$$\min(\tfrac{1}{4}R) = -\tfrac{1}{2}(1-\surd\rho).$$

When $\rho = 3-2\surd 2$ this gives $\min(\tfrac{1}{4}R) = -(1-\tfrac{1}{2}\surd 2) = -0{\cdot}293$; when $\rho = 1$, $\min(R) = 0$, while for $\rho = 3+2\surd 2$, $\min(R) = \tfrac{1}{2}\surd 2 = 0{\cdot}707$. The origin of the Type V curve will in this case be between $X = -0{\cdot}707$ and $X = +0{\cdot}293$, or $x = -1{\cdot}207$ and $x = -0{\cdot}207$.

There is a discontinuity in $\min(\tfrac{1}{4}R)$, when $\rho = 3-2\surd 2$ for, as ρ (increasing) passes through $3-2\surd 2$, $\min(a+k)$ changes from 0 to $2(\surd 2-1)$ and $\min(\tfrac{1}{4}R)$ changes from $-\tfrac{1}{2}\surd 2$ to $(\tfrac{1}{2}\surd 2-1)$ (see Fig. 1).]

PART III. APPENDIX

A. *The case when a, k are complex conjugate*

The Generalized Waring Distribution still has real frequencies if a, k are complex conjugate. It is not proposed to discuss this case in detail in this paper. It may be noted, however, that if $a = \gamma+i\delta$, $k = \gamma-i\delta$,

$$\left.\begin{aligned} f_r &= \frac{c\{\gamma^2+\delta^2\}\{(\gamma+1)^2+\delta^2\}\ldots\{(\gamma+r+1)^2+\delta^2\}}{(\rho+2\gamma)(\rho+2\gamma+1)\ldots(\rho+2\gamma+r-1)\,r!}, \\ &\text{where} \\ c &= \frac{\Gamma(\rho+\gamma+i\delta)\,\Gamma(\rho+\gamma-i\delta)}{\Gamma(\rho)\,\Gamma(\rho+2\gamma)} \\ &= \prod_{n=0}^{\infty}\frac{(\rho+n)(\rho+2\gamma+n)}{(\rho+\gamma+n)^2+\delta^2}. \end{aligned}\right\} \tag{1}$$

Let us assume that γ is positive; otherwise infinite or negative frequencies might occur. δ may be taken as positive without loss of generality, since if it is negative a and k may be interchanged. Then $(a+k)$ and ak are both positive. We have seen that the roots of the fundamental quadratic in the denominator of (8.7) are real if

$$\{a+\tfrac{1}{2}(\rho-1)\}^2\{k+\tfrac{1}{2}(\rho-1)\}^2 \geqslant \tfrac{1}{4}(\rho+1)^2. \tag{2}$$

Now let

$$a' = (\gamma'+\delta')/\surd 2, \quad k' = (\gamma'-\delta')/\surd 2, \tag{3}$$

where, when a and k are real, $\gamma=(a+k)/2$, $\delta=(a-k)/2$, $\gamma'=\sqrt{2}\gamma$, $\delta'=\sqrt{2}\delta$; and when a and k are conjugate complex,

$$a=\gamma+i\delta,\quad k=\gamma-i\delta,\quad \delta>0,\quad \gamma=\tfrac{1}{2}(a+k),$$
$$i\delta=\tfrac{1}{2}(a-k),\quad \gamma^1=\sqrt{2}\gamma,\quad \delta'=\sqrt{2}\delta=-i\sqrt{2}(a-k)|2.$$

When a,k are real, the relation (2) (corresponding to the area of the quadrant outside the circle) becomes

$$\left\{\gamma'+\frac{1}{\sqrt{2}}(\rho-1)\right\}^2+\delta'^2\geqslant\tfrac{1}{4}(\rho+1)^2. \tag{4}$$

When a and k are complex conjugate, δ'^2 in (4) becomes $-\delta'^2$. The equation

$$\left\{\gamma'+\frac{1}{\sqrt{2}}(\rho-1)\right\}^2-\delta'^2=\tfrac{1}{4}(\rho+1)^2 \tag{5}$$

represents a rectangular hyperbola with asymptotes $\{\gamma'+[(\rho-1)/\sqrt{2}]\}\pm\delta'=0$ and centre at $(-(\rho-1)/\sqrt{2},0)$ or at $a'=k'=-\frac{1}{2}(\rho-1)$. Alternatively, the hyperbola may be written

$$2\{a'+\tfrac{1}{2}(\rho-1)\}\{k'+\tfrac{1}{2}(\rho-1)\}=\tfrac{1}{4}(\rho+1)^2. \tag{5 bis}$$

The "vertices" of the hyperbola (5) are at $\gamma'=-(\rho-1)\sqrt{2}\pm\frac{1}{2}(\rho+1)$, $\delta'=0$ or $a'=k'=-\frac{1}{2}(\rho-1)\pm\frac{1}{4}\sqrt{2}(\rho+1)$.

When $0<\rho\leqslant 3-2\sqrt{2}$, both branches of the hyperbola have portions within the positive quadrant. When $3-2\sqrt{2}<\rho\leqslant 3+2\sqrt{2}$, only one branch intersects the positive quadrant. When $\rho>3+2\sqrt{2}$ neither branch intersects the positive quadrant.

The roots of the fundamental quadratic are real if the point (γ',δ') or (a',k') lies on the opposite side of the hyperbola from its centre; they are equal if the point is on the hyperbola and complex otherwise. The continuous analogue will be Type VI, Type V or Type IV accordingly.

The hyperbolas given by (5) have their centres at the same points as the circular contours of Fig. 1; and touch the latter along the diagonal OC; also the "radii" (semi-major axes in the case of the hyperbolas) are the same—namely $\frac{1}{2}(\rho+1)$. When $\rho>3+2\sqrt{2}$, the hyperbolas are entirely outside the first quadrant, and the continuous analogue is Type VI, as before. Everything proceeds as in Fig. 1, with the circular contours replaced by hyperbolic contours; except that the unit square has now no particular relevance.

The results are illustrated in Fig. 3. For convenience, except at O, only the portions of the hyperbolas within the square $OA'B'C'$ of side 2 units, are shown. For any given value of $\rho\geqslant 0{\cdot}172$ the corresponding hyperbola divides the positive quadrant into two portions. Above it the points correspond to values of a,k, ρ for which the continuous analogue of the G.W. distribution is Type VI; below it Type IV, on it to Type V. For $0<\rho<0{\cdot}172$, both branches of the hyperbolas intersect the positive quadrant; consequently there is also a small Type VI area in the positive quadrant close to O.

B. *General note*

It seems worth while noting that we must have $|\rho|>0$; otherwise the hypergeometric would not converge. But if, while keeping a,k real, we allow one or both of them to be negative, the method employed here might be used to classify all the

Pearson Types according to the values of the parameters of the hypergeometric from which they are derived. This, however, would be quite beyond the scope or intention of this paper.

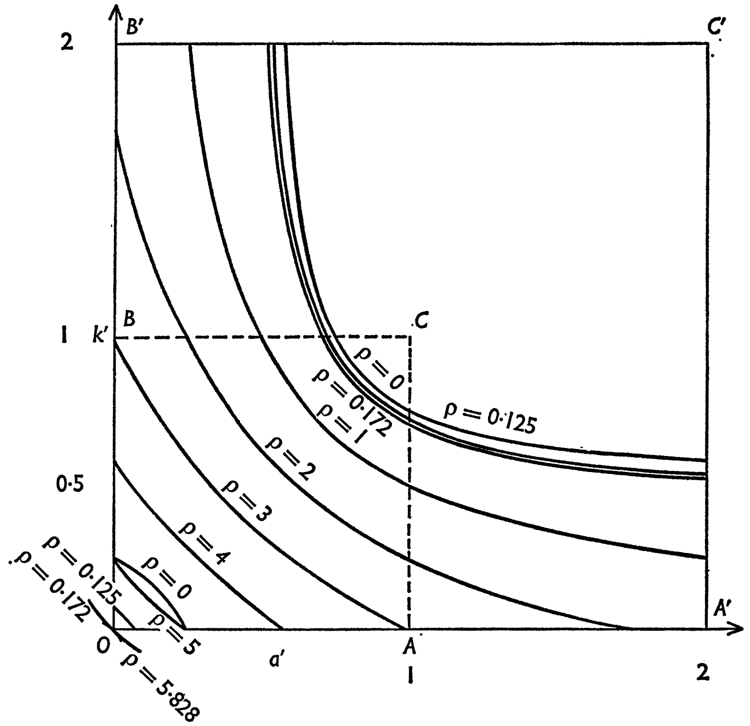

FIG. 3. Hyperbolas of the system $2\{a'+\frac{1}{2}(\rho-1)\}\{k'+\frac{1}{2}(\rho-1)\} = \frac{1}{4}(\rho+1)^2$. Here $a = \gamma+i\delta$, $k = \gamma-i\delta$, $a' = \surd 2a$, $k' = \surd 2k$ where a, k are the parameters of the G.W.D. For given ρ, the continuous analogue of the G.W.D. is Pearson's Type VI if the point a', k' is in the first quadrant and above the upper branch of the hyperbola. It is Type IV if the point is between the two branches of the hyperbola, and Type V if the point is on the hyperbola. It is only for $0 < \rho < 0{\cdot}172$ that the lower branch intersects the positive quadrant. For other values of ρ it is irrelevant.

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