CHANCE MECHANISMS FOR THE UNIVARIATE GENERALIZED WARING DISTRIBUTION AND RELATED CHARACTERIZATIONS

EVDOKIA XEKALAKI*

Department of Statistics
University of Dublin
Trinity College, IRELAND

SUMMARY. The intent of this paper is to provide an anthology of results on the subject of models (chance mechanisms) that give rise to the Univariate Generalized Waring Distribution. These include results that have appeared in the statistical literature before as well as some new ones that appear for the first time in this paper. Some characterization problems relating to certain genesis schemes are also considered.

KEY WORDS. Univariate generalized Waring distribution, urn models, conditionality models, STER model, coin-tossing game model, characterization.

1. INTRODUCTION

The Univariate Generalized Waring Distribution with parameters $a > 0$, $k > 0$ and $\rho > 0$ (UGWD$(a,k;\rho)$) is the distribution whose probability generating function (p.g.f.) is given by

$$G(s) = \frac{\rho_{(k)}}{(a+\rho)_{(k)}}\ {}_2F_1\ (a,k;\ a+k+\rho;\ s) \qquad (1)$$

where $\alpha_{(\beta)} = \Gamma(\alpha+\beta)/\Gamma(\alpha)$ for any complex numbers α, β and ${}_2F_1$ is the Gauss hypergeometric series obtained as a special case of

*Present address: Department of Statistics, University of Missouri-Columbia, Columbia, MO 65211, USA.

C. Taillie et al. (eds.), Statistical Distributions in Scientific Work, Vol. 4, 157–171.

$$ {}_pF_q(a_1,\cdots,a_p;\ b_1,\cdots,b_q;\ s) = \sum_{r=0}^{\infty} \frac{(a_1)_{(r)} \cdots (a_p)_{(r)}}{(b_1)_{(r)} \cdots (b_q)_{(r)}} \frac{s^r}{r!} $$

for $p = 2$, $q = 1$. (2)

If we write

$$ k = -n, \quad a = -Np, \quad \rho = N + 1 $$

then (1) becomes the familiar generalized hypergeometric distribution

$$ \frac{(N-n)!}{N!} \frac{(Nq)!}{(Nq-n)!} \ {}_2F_1(-n, -Np; Nq-n+1; s), \qquad q = 1-p $$

where $z! = \int_0^{\infty} e^{-t} t^z \, dt$ for all real z and $\frac{(-x)!}{(-x-y)!} = (-1)^y \frac{(x+y-1)!}{(x-1)!}$ for integer y. (See Jordan, 1927; Davies, 1933, 1934; Kemp and Kemp, 1956; Sarkadi, 1957; Kemp, 1968a; Shimizu, 1968; Dacey, 1969; Janardan and Patil, 1972; Sibuya and Shimizu, 1980a,b).

The name generalized Waring was given to this distribution by Irwin (1963) who based its derivation on a generalization of Waring's formula. In the 18*th* century, Waring showed that the function $\frac{1}{x-a}$, $x > a$, can be expanded in the following way

$$ \frac{1}{x-a} = \sum_{r=0}^{\infty} \frac{a_{(r)}}{x_{(r+1)}} . $$

Irwin extended this formula by showing that

$$ \frac{1}{(x-a)_{(k)}} = \sum_{r=0}^{\infty} \frac{a_{(r)}\, k_{(r)}}{x_{(r+k)}} \frac{1}{r!}, \qquad x > a > 0, \quad k > 0. $$

Multiplying both sides by $\rho = x-a$ he ended up with a series which converged to unity. The successive terms of this series were then considered by him as defining a discrete probability distribution which he called the generalized Waring distribution. For certain values of the parameters the UGWD($a,k;\rho$) can be very long-tailed and so it was shown (Irwin, 1963, 1975) to be a suitable theoretical form for the description of biological distributions. Actually the UGWD($a,k;\rho$) showed an improvement as compared to its particular case, the simple Waring ($k=1$), which was also used for the same type of data (Irwin, 1963). It is interesting that another special case of the distribution when $a = k = 1$ was obtained by Yule (1924) also on a biological hypothesis. The latter case, i.e., the UGWD $(1,1; \rho)$ was later called the Yule distribution by Kendall (1961) who suggested it for bibliographic and economic applications. Both, the simple Waring and the Yule distributions were considered by various authors for describing word frequency data, e.g., Simon (1955, 1960), Haight (1966), Herdan (1964). Another very important application of the UGWD$(a,k; \rho)$ was considered by Irwin (1968, 1975) who suggested it as a theoretical model for accident distributions in the context of accident proneness. Compared with the negative binomial, the UGWD provided a better fit. But, as stressed by Irwin the importance of this model lies in that it enables us to partition the variance into separate additive components due to proneness, risk exposure, and randomness; thus by fitting it we can infer about the role that each of these factors has played in a given accident situation. One would, therefore, be interested in the underlying chance mechanisms that lead to the UGWD.

So, the subsequent sections attempt to draw together various existing results concerning the genesis of this distribution, suggest some new genesis schemata and prove certain characterization theorems connected with them.

2. URN MODELS

Consider an urn containing 'a' white and 'b' black balls. One ball is drawn at random and replaced along with 1 additional ball of the same color before the next ball is drawn. The process is repeated until 'k' white balls are drawn. The number X of black balls drawn before the kth white ball has the UGWD$(b,k; a)$, i.e.,

$$P(X = x) = \frac{a_{(b)}}{(a+k)_{(b)}} \cdot \frac{b_{(x)} k_{(x)}}{(a+b+k)_{(x)}} \frac{1}{x!}$$

(Jordan, 1927; Kemp and Kemp, 1956; Sarkadi, 1957; Dacey, 1969; Johnson and Kotz, 1977).

Clearly, this is a special case of Polya's inverse urn scheme where each ball drawn is replaced with c additional balls of the same color. Hence, when the parameters of the UGWD are positive integers, the distribution can be considered as a special case of the inverse Polya distribution, for $c = 1$.

An alternative urn representation of the UGWD may be obtained from the following generalization of Friedman's (1949) inverse urn scheme. Consider an urn containing 'a' white balls and 'b' black balls. One ball is drawn at random and replaced by $1+\alpha$ balls of the same color along with β balls of the opposite color. Drawings are continued until k black balls are drawn. The number X of white balls drawn before the k*th* black ball has a frequency distribution given by

$$P(X = x) = \left(\frac{\alpha}{\alpha+\beta}\right)^{k+x} \frac{\left(\frac{a}{\alpha}\right)^{(x + k\beta/\alpha)}}{\left(\frac{a + b}{\alpha + \beta}\right)^{(k + x)}} \times$$

$$\sum_{x_{k-1}=0}^{x_k} \cdots \sum_{x_1=0}^{x_2} \prod_{r=1}^{k} \frac{\frac{b}{\alpha} + \frac{\beta}{\alpha} x_r + r - 1}{\left(\frac{a}{\alpha} + x_r + (r-1)\frac{\beta}{\alpha}\right)^{(\beta/\alpha)}}, \quad x_k \equiv x. \qquad (3)$$

If we let $\beta = 0$, (3) reduces to the UGWD$(k, \frac{a}{\alpha}; \frac{b}{\alpha})$. (Note that when $k=1$ the urn scheme considered reduces to Friedman's inverse scheme which (Kemp, 1968a) gives rise to the UGWD$(1, \frac{a}{\alpha}; \frac{b}{\alpha})$.)

3. MIXED MODELS

3.1 The UGWD as a Mixture of Negative Binomial, Poisson and Generalized Poisson Distributions. Let X be a random variable (r.v.) having the negative binomial distribution with parameters k and Q. Let $g(s)$ be its p.g.f., i.e.,

$$g(s) := [1 + Q(1-s)]^{-k}, \qquad Q > 0, \quad k > 0. \qquad (4)$$

Let Q follow a beta distribution of the second kind with parameters a and ρ, i.e.,

$$f(Q) = \frac{\Gamma(a + \rho)}{\Gamma(a)\,\Gamma(\rho)} Q^{a-1} (1+Q)^{-(a+\rho)}, \qquad Q > 0, \quad a > 0, \quad \rho > 0. \qquad (5)$$

Then the distribution of X has p.g.f. given by

$$G(s) = \frac{\Gamma(a + \rho)}{\Gamma(a)\Gamma(\rho)} \int_0^\infty (1 + Q(1-s))^{-k} Q^{a-1} (1+Q)^{-(a+\rho)} dQ,$$

which is the well-known integral representation of the Gauss hypergeometric function (e.g. Erdélyi *et al.*, 1953), i.e.,

$$G(s) = \frac{\rho_{(k)}}{(a+\rho)_{(k)}}\ {}_2F_1(a,k;\ a+k+\rho;\ s) \sim \text{UGWD}(a,k;\ \rho)$$

(Irwin, 1968). By the transformation $q = Q/(1+Q)$ (4) and (5) take the forms

$$g(s) = (1-q)^k(1-qs)^{-k}, \qquad 0 < q < 1,\ \ k > 0 \qquad (6)$$

and

$$h(q) = \frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)}\, q^{a-1}\, (1-q)^{\rho-1}, \qquad a,\rho > 0,\ \ 0 < q < 1 \quad (7)$$

respectively.

Then the UGWD$(a,k;\ \rho)$ results as the mixture on q of the negative binomial distribution as given by (6) if q is a r.v. having the beta distribution of the first type with parameters a and ρ and probability density function (p.d.f.) given by (7) (Kemp and Kemp, 1956; Sarkadi, 1957; Irwin, 1968; Janardan, 1973).

The derivation of the negative binomial as a gamma mixture of the Poisson distribution or as a Poisson generalized by a logarithmic series distribution indicates that the UGWD can also arise from the following models.

$$\text{Poisson}(\lambda)\underset{\lambda}{\wedge}\ \text{gamma}(a;\ b^{-1})\underset{b}{\wedge}\ \text{beta II}\ (k;\rho) \sim \text{UGWD}(a,k;\ \rho) \quad (8)$$

$$\text{Poisson}(\lambda)\underset{\lambda}{\wedge}\ \text{gamma}(a;\ b^{-1})\underset{b/1+b}{\wedge}\ \text{beta I}\ (k;\rho) \sim \text{UGWD}(a,k;\ \rho) \quad (9)$$

$$\text{Poisson}(-\lambda\log(1-\theta))\vee\text{log series}(\theta)\underset{\theta}{\wedge}\text{beta I}(a;\rho) \sim \text{UGWD}(a,\lambda;\ \rho) \quad (10)$$

$$\text{Poisson}(\lambda\log(1+\theta))\vee\text{log series}\left(\frac{\theta}{1+\theta}\right)\underset{\theta}{\wedge}\text{beta II}\ (a;\rho) \sim \text{UGWD}(a,\lambda;\ \rho) \quad (11)$$

Models (8) and (9) were considered by Irwin (1968) who gave λ and b an accident liability and accident proneness interpretation respectively to obtain the UGWD as the underlying accident distribution.

Another interesting mixed Poisson model was considered by Dacey (1969) in the context of a problem in geographical analysis.

Let X be a discrete r.v. having the Poisson distribution with parameter λ, $\lambda > 0$. Assume that λ is itself a r.v. with some distribution function $F(\lambda)$ such that

$$dF(\lambda) = \frac{\Gamma(c-b)\Gamma(c-a)}{\Gamma(b)\Gamma(c-a-b)\Gamma(a)} e^{\lambda/2} \lambda^{(a+b-3)/2} W_{\mu,\nu}(\lambda)d\lambda,$$

$$\mu = \frac{1}{2}(a+b+1)-c, \quad \nu = \frac{1}{2}(a-b).$$

Here $W_{\mu,\nu}(\lambda)$ denotes the Whittaker function identified by the integral equation

$$W_{\mu,\nu}(\lambda) = \frac{e^{-\lambda/2} \lambda^{\nu+1/2}}{\Gamma(\frac{1}{2}+\nu-\mu)} \int_0^\infty e^{-\lambda t} t^{\nu-\mu-1/2} (1+t)^{\mu+\nu-1/2} dt,$$

$$\nu + \frac{1}{2} > \mu, \quad \lambda > 0.$$

Hence

$$P(X = r) = \int_0^\infty e^{-\lambda} \frac{\lambda^r}{r!} dF(\lambda)$$

$$= \frac{(c-a-b)_{(b)}}{(c-b)_{(b)}} \frac{a_{(r)}\, b_{(r)}}{c_{(r)}} \frac{1}{r!}, \quad a,b > 0, \quad c > a+b.$$

But, this is the probability function (p.f.) of the UGWD(a,b; c-a-b)

3.2 The UGWD as a Mixed Confluent Hypergeometric Distribution. Bhattacharya (1966) obtained the negative binomial distribution with parameters b and a^{-1} by compounding a distribution with p.g.f. of the form

$$g(s) = \frac{{}_1F_1(b;d;\lambda s)}{{}_1F_1(b;d;\lambda)}, \quad \lambda > 0, \tag{12}$$

with a continuous distribution belonging to what he called a 'generalized exponential family' with p.d.f.

$$f(\lambda) = \frac{1}{\Gamma(d)} a^b (a+1)^{d-b} \lambda^{d-1} e^{-(a+1)\lambda} {}_1F_1(b;d;\lambda), \quad \lambda,b,d,a > 0. \tag{13}$$

Here ${}_1F_1$ is the confluent hypergeometric series given by (1) for $p = q = 1$.

The class of distributions defined by (12) contains many known distributions such as the hyper-Poisson for $b = 1$ (Bardwell and Crow, 1964) and the Poisson $\wedge$ tail-truncated gamma distribution for $a = b+1$ (Kemp, 1968b). On the other hand, Bhattacharya's (13) family includes the gamma $(d;a)$ for $d=b$ and consequently it, also, includes the exponential and the chi-square distributions as special cases.

More generally, Kemp and Kemp (1971) showed that distributions with p.g.f.'s of the form

$$g(t) = \frac{{}_2F_1(b,c;\ d;\ s(a+1)^{-1})}{{}_2F_1(b,c;\ d;\ (a+1)^{-1})}, \qquad a \geq 0, \tag{14}$$

result as the mixture on the parameter λ of a distribution of family (12), if λ is a r.v. having a distribution with p.d.f.

$$f(\lambda) = \frac{(a+1)^c \lambda^{c-1} e^{-(a+1)\lambda}\ {}_1F_1(b;\ d;\lambda)}{\Gamma(c)\,{}_2F_1(b,\ c;\ d;\ (a+1)^{-1})}, \quad \begin{matrix} \lambda > 0 \\ a \geq 0 \\ c > 0 \end{matrix} \tag{15}$$

The latter family includes the gamma(c;a) distribution as a special case and hence the exponential and the chi-square distributions.

The UGWD(b,c ; ρ) belongs to the family (14) for $\rho = d-b-c$, $b,c > 0$ and $a = 0$. Hence, following Kemp and Kemp's argument we can obtain the UGWD(b,c; d-b-c) as a mixture on λ of a distribution belonging to (15) if λ has a distribution with p.d.f.

$$f(\lambda) = e^{-\lambda}\lambda^{c-1} \frac{{}_1F_1(b;\ d;\ \lambda)}{{}_2F_1(b,c;\ d;\ 1)\Gamma(c)}, \quad \lambda > 0;\ a,\ b,\ c > 0 \tag{16}$$

provided that $d-b-c > 0$. Thus,

$$\int_0^\infty \frac{{}_1F_1(b;\ d;\ \lambda s)}{{}_1F_1(b;\ d;\ \lambda)}\, e^{-\lambda}\lambda^{c-1} \frac{{}_1F_1(b;\ d;\ \lambda)}{{}_2F_1(b,c;\ d;\ 1)\Gamma(c)}\, d\lambda \sim \text{UGWD}(b,c;\ d-b-c).$$

4. CONDITIONALITY MODELS

In this section we consider certain new derivations of the UGWD based on what we term conditionality models. These are, in fact, mixed models with discrete mixing distribution.

Model 4.1. Let X and Y be non-negative discrete r.v.'s such that the conditional distribution of Y given (X = x) is the negative hypergeometric with parameters x, m and N and p.f. given by

$$P(Y=y|X=x) = \frac{\binom{-m}{y}\binom{-N+m}{x-y}}{\binom{-N}{x}}, \quad m,\ N > 0, \quad y = 0,1,\cdots,x. \tag{17}$$

Let the distribution of X be the UGWD $(a,N;\rho)$. Then the distribution of Y is the UGWD $(a,m;\rho)$. To prove this, we substitute for $P(X=x)$ and $P(Y=y|X=x)$ in the well-known formula

$$P(Y=y) = \sum_x P(Y=y|X=x)\ P(X=x) \tag{18}$$

and obtain

$$P(Y=y) = \frac{\rho_{(N)}}{(a+\rho)_{(N)}} \frac{m_{(y)}a_{(y)}}{(a+N+\rho)_{(y)}} \frac{1}{y!} \sum_{x=0}^{\infty} \frac{(N-m)_{(x)}(a+y)_{(x)}}{(a+N+y+\rho)_{(x)}} \frac{1}{x!}$$

$$= \frac{\rho(N)^{m}(y)^{a}(y)}{(a+\rho)_{(N+y)}} \frac{1}{y!} \frac{(\rho+m+a+y)_{(N-m)}}{(\rho+m)_{(N-m)}}$$

$$= \frac{\rho_{(m)}}{(a+\rho)_{(m)}} \frac{a_{(y)}m_{(y)}}{(a+m+\rho)_{(y)}} \frac{1}{y!}$$

which establishes the result.

Hence, the UGWD is reproducible with respect to (w.r.t.) the negative hypergeometric family of distributions (in Skibinsky's (1970) terminology). Note that, for certain limiting values of the parameters, the UGWD tends to the negative binomial distribution (Irwin, 1975) which also enjoys this property. It is interesting, therefore, to observe that reproducibility w.r.t. the negative hypergeometric family is preserved under the passage from the UGWD to the negative binomial limit.

It is also interesting to point out here that the converse of this result is also true, i.e.,

$$\text{if } Y \sim \text{UGWD}(a,m;\rho) \text{ then } X \sim \text{UGWD}(a,N;\rho). \tag{19}$$

To show this we use the following lemma.

Lemma. The family of negative hypergeometric distributions with p.f. $P_x = \binom{-m}{x}\binom{-N+m}{n-x}\Big/\binom{-N}{n}$, $m, N, n > 0$, $x = 0,1,\cdots,n$ is complete w.r.t. the parameter m.

It can now be seen that (18) is a functional equation in $P(X = x)$ where $Y \sim \text{UGWD}(a,m;\rho)$. One solution is the UGWD$(a,N;\rho)$ which because of the lemma is unique.

The above discussion leads us to the following characterization theorem.

Theorem 1. Let X and Y be non-negative, integer-valued r.v.'s such that the conditional distribution of Y given (X=x) is the negative hypergeometric with parameters x, m and N as given by (17). Then the distribution of X is the UGWD(a,N; ρ) if and only if (iff) the distribution of Y is the UGWD(a,m; ρ).

Consider now the following model.

Model 4.2. Let X, Y be two r.v.'s such that the conditional distribution of $X|(Y=y)$ is the UGWD(a+y, n; ρ+m), n,m > 0 shifted y units to the right. Let the distribution of Y be the UGWD(a,m; ρ). Then, the distribution of X is the UGWD (a,m+n; ρ). To prove this let G(t) denote the p.d.f. of X. Then

$$G(t) = \rho_{(m+n)} \sum_y \frac{t^y {}_2F_1(a+y,n;\ a+y+n+\rho+m;\ t)\ a_{(y)} m_{(y)}}{(a+\rho)_{(m+n+y)}\ y!}$$

$$= \frac{\rho_{(m+n)}}{(a+\rho)_{(m+n)}} \sum_x \sum_y \frac{a_{(x+y)} n_{(x)} m_{(y)}}{(a+\rho+m+n)_{(x+y)}} \frac{t^x}{x!} \frac{t^y}{y!}$$

$$= \frac{\rho_{(m+n)}}{(a+\rho)_{(m+n)}} \sum_x \frac{a_{(x)} (m+n)_{(x)}}{(a+\rho+m+n)_{(x)}} \frac{t^x}{x!}$$

Hence the distribution of X is the UGWD(a,m+n; ρ).

The converse of this result is not true in general. It holds, however, when a=1, i.e., if the distribution of X is the UGWD(1,m+n; ρ) then the distribution of Y is the UGWD(1,m; ρ). This can be shown by an argument similar to that employed in Theorem 1.

This provides the following characterization theorem.

Theorem 2. Let X, Y be two r.v.'s such that the conditional distribution of $X|(Y=y)$ is the UGWD(y+1,n; ρ+m),ρ,n,m > 0 shifted y units to the right. Then the distribution of X is the UGWD(1,m+n; ρ) iff the distribution of Y is the UGWD(1,m; ρ).

5. THE IDEAL COIN-TOSSING GAME MODEL

In this section we suggest another new genesis scheme, arising from a fair coin-tossing game. Consider a gambler, say A, who at each trial wins or loses a unit amount and let S_N

denote A's cumulative gain in 2N independent trials. This gambling game can be interpreted as the record of an ideal experiment which consists of 2N successive tosses of a coin. Let

$$X_j = \begin{cases} 1 & \text{if "heads" at the } j\text{th trial} \\ -1 & \text{if "tails" at the } j\text{th trial.} \end{cases} \tag{20}$$

Obviously, $P(X_j=1) = P(X_j=-1) = \frac{1}{2}$, $j = 1,2,\cdots,2N$.

Then $S_k = X_1 + X_2 + \cdots + X_{2k}$, $k = 1,2,\cdots,N$.

and

$$P(S_k = 0) = \binom{2k}{k} 2^{-2k}, \quad k = 0,1,2,\cdots,N$$

(see Feller, 1968, p. 273).

Suppose now that N is not a fixed number. Assume, instead, that N is a r.v. and let its distribution be the UGWD(1,a; ρ). Then

$$P(N=r|S_N=0) = \frac{P(S_N=0|N=r)P(N=r)}{\sum_{r=0}^{\infty} P(S_N=0|N=r)P(N=r)}$$

$$= \frac{\binom{2r}{r}2^{-2r}a_{(r)}/(a+\rho+1)_{(r)}\ r!}{\sum_{r=0}^{\infty}\binom{2r}{r}2^{-2r}a_{(r)}/(a+\rho+1)_{(r)}\ r!}$$

$$= \frac{(\rho+\frac{1}{2})_{(a)}}{(\rho+1)_{(a)}} \frac{(\frac{1}{2})_{(r)}\ a_{(r)}}{(a+\rho+1)_{(r)}} \frac{1}{r!}$$

$$\sim \text{UGWD}(\tfrac{1}{2}, a; \rho+\tfrac{1}{2})$$

Hence, if N is a UGWD(1,a; ρ) r.v. then N given a total gain of 0 is a UGWD($\frac{1}{2}$,a; ρ+$\frac{1}{2}$) r.v. (The case a = ρ = 1 has been examined by Shimizu, 1968).

The converse of the above result can easily be shown to hold. Therefore, the following characterization theorem can be established.

Theorem 3. Let X_j, $j = 1,2,\cdots,2N$, be defined as in (20). Let N be a non-negative integer-valued r.v. and let S_N denote the random sum $X_1+X_2+\cdots+X_{2N}$. Then, the distribution of N is the UGWD(1,a; ρ) iff the distribution of $N|(S_N=0)$ is the UGWD $(\frac{1}{2}, a; \rho+\frac{1}{2})$.

6. THE "STER" MODEL

Bissinger (1965) observed that in a great many inventory decision problems, the frequency distribution defined by

$$q_y = \frac{1}{1-p_0} \sum_{x=y+1}^{\infty} \frac{p_x}{x}, \qquad y = 0,1,2,\cdots \tag{21}$$

arises, where p_x is the probability function of the demand r.v. X. Here, the probabilities q_y are defined as Sums successively Truncated from the Expectation of the Reciprocal of the variable X (STER). Xekalaki (1980) showed that under certain conditions q_y may be thought of as interpreting the fluctuations of the stock in hand, say Y and proved that the distribution of the demand, X is the left-truncated UGWD (1,1; ρ) at the point $k-1$ iff $X \overset{d}{=} Y|(Y \geq k)$. (The case $k=1$ has been examined by Krishnaji, 1970.)

It follows then, that (for $k=0$) the STER model in (21) gives rise to the UGWD(1,1; ρ) iff the r.v.'s X and Y are identically distributed.

7. MISCELLANEOUS DERIVATIONS

Consider the Kolmogorov differential equations for the birth-and-death process:

$$\frac{dP_0(t)}{dt} = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

$$\frac{dP_n(t)}{dt} = -(\mu_n+\lambda_n)P_n(t) + \lambda_{n-1}P_{n-1}(t)+\mu_{n+1}P_{n+1}(t), \qquad n \geq 1$$

where λ_n and μ_n are the birth and death rates of the *nth* state, respectively. Kemp and Kemp (1975) obtained a generalized hypergeometric form for the equilibrium distribution by suitably defining the ratio λ_{n-1}/μ_n. In particular, assuming that

$$\frac{\lambda_{n-1}}{\mu_n} = \frac{(a+n-1)(b+n-1)}{(c+n-1)n} \tag{22}$$

their resultant equilibrium distribution had p.g.f. given by

$$C\ {}_2F_1(a,b;\ c;\ s) \tag{23}$$

where C is the normalizing constant. Clearly (23) can be the p.g.f. of the UGWD provided that $c > a+b$. That is, if in (22) c is chosen so that $a+b < c$ then the equilibrium solution given by (23) is the UGWD(a,b; c-a-b).

Let us now mention two further models that generate some special forms of the UGWD. How these can be extended so as to give rise to the general form of the UGWD remains an open problem.

Kemp and Kemp (1968) examined the distribution with p.g.f.

$$G_a(s) = s^a p^a\ {}_2F_1(\tfrac{1}{2}a,\ \tfrac{1}{2}(a+1);\ a+1;\ 4pqs), \qquad p \geq q, \quad a > 0 \tag{24}$$

which they termed "the lost games distribution." Clearly, for $p=q=\frac{1}{2}$ this reduces to a UGWD$(\frac{a}{2}, \frac{a+1}{2}; \frac{1}{2})$ shifted 'a' units to the right. They obtained (24) (i) as the distribution of the total number of games lost by the ruined ~~gambler starting with~~ 'a' monetary units against an infinitely rich adversary and (ii) as the distribution of the number of customers served in a busy period (starting with 'a' customers) of an M/M/1 queue.

Finally, Shimura and Takahasi (1967) discuss a genesis scheme of the UGWD(1,1; 1) in connection with a problem in branching processes.

ACKNOWLEDGEMENTS

I would like to thank Professors G. P. Patil, S. Kotz, R. Shimizu and K. G. Janardan for interesting discussions we had during the summer school which helped in the improvement of an earlier version of this paper. Particularly, I would like to express my appreciation to Professor R. Shimizu for bringing some references to my attention and for translating two of them. The comments of the referees are also appreciated.

This research was supported in part by a grant from the "Alexandros Onasis" Foundation.

REFERENCES

Bardwell, G. E. and Crow, E. L. (1964). A two parameter family of hyper-Poisson distributions. *Journal of the American Statistical Association,* 59, 133-141.

Bhattacharya, S. K. (1966). Confluent hypergeometric distributions of discrete and continuous type with applications to accident proneness. *Bulletin of the Calcutta Statistical Association,* 15, 20-21.

Bissinger, B. H. (1965). A type resisting distribution generated from considerations of an inventory decision model. In *Classical and Contagious Discrete Distributions,* G. P. Patil, ed. Pergamon Press and Statistical Publishing Society, Calcutta, Pages 15-17.

Dacey, M. F. (1969). A hypergeometric family of discrete probability distributions: Properties and applications to location models. *Geographical Analysis,* 1, 283-317.

Davies, O. L. (1933). On asymptotic formulae for the hypergeometric series. *Biometrika,* 25, 295-322.

Davies, O. L. (1934). On asymptotic formulae for the hypergeometric series. *Biometrika,* 26, 59-107.

Erdélyi, A. *et al.* (1953). *Higher Transcendental Functions, Vol. 1.* McGraw-Hill, New York.

Feller, W. (1968). *An Introduction to Probability Theory and its Applications, Vol. 1.* Wiley, New York.

Friedman, B. (1949). A simple urn model. *Communications on Pure and Applied Mathematics,* 2, 59-70.

Haight, F. A. (1966). Some statistical problems in connection with word association data. *Journal of Mathematical Psychology,* 3, 217-233.

Herdan, G. (1964). *Quantitative Linguistics.* Butterworths, London.

Irwin, J. O. (1963). The place of mathematics in medical and biological statistics. *Journal of the Royal Statistical Society, Series A,* 126, 1-44.

Irwin, J. O. (1968). The generalized Waring distribution applied to accident theory. *Journal of the Royal Statistical Society, Series A,* 131, 205-225.

Irwin, J. O. (1975). The generalized Waring distribution. *Journal of the Royal Statistical Society, Series A,* 138, 18-31 (Part I), 204-227 (Part II), 374-384 (Part III).

Janardan, K. G. (1973). Chance mechanisms for multivariate hypergeometric models. *Sankhyā, Series A,* 35, 465-478.

Janardan, K. G. and Patil, G. P. (1972). A unified approach for a class of multivariate hypergeometric models. *Sankhyā, Series A,* 34, 363-376.

Johnson, N. L. and Kotz, S. (1977). *Urn Models and Their Application.* Wiley, New York.

Jordan, C. (1927). Sur un cas généralisé de la probabilité des épreuves répétées. *Acta Scientiarum Mathematicarum,* 3, 193-210.

Kemp, A. W. (1968a). A wide class of discrete distributions and the associated differential equations. *Sankhyā, Series A*, 30, 401-410.

Kemp, A. W. (1968b). A limited risk cPp. *Skandinavisk Aktuarietidskrift*, 51, 198-203.

Kemp, A. W. and Kemp, C. D. (1968). On a distribution associated with certain stochastic processes. *Journal of the Royal Statistical Society, Series B*, 30, 160-163.

Kemp, A. W. and Kemp, C. D. (1971). On mixing processes and the lost-games distribution. *Zastosowania Matematyki*, 12, 167-173.

Kemp, A. W. and Kemp, C. D. (1975). Models for Gaussian hypergeometric distributions. In *Statistical Distributions in Scientific Work, Vol. 1*, G. P. Patil, S. Kotz and J. K. Ord, eds. Reidel, Dordrecht-Holland. Pages 31-40.

Kemp C. D. and Kemp, A. W. (1956). Generalized hypergeometric distributions. *Journal of the Royal Statistical Society, Series B*, 18, 202-211.

Kendall, M. G. (1961). Natural law in the social sciences. *Journal of the Royal Statistical Society, Series A*, 124, 1-16.

Krishnaji, N. (1970). A characteristic property of the Yule distribution. *Sankhyā, Series A*, 32, 343-346.

Sarkadi, K. (1957). Generalized hypergeometric distributions. *A Magyar Tudományos Akadémia Matematikai Kutató Intézet Közleményei*, 2, 59-69.

Shimizu, R. (1968). Generalized hypergeometric distributions. *Proceedings of the Institute of Statistical Mathematics*, 16, 147-165 (in Japanese).

Shimura, T. and Takahasi, K. (1967). On the moments and the examples of the distribution of the time to extinction in the Galton-Watson process. *Proceedings of the Institute of Statistical Mathematics*, 15, 161-166 (in Japanese)

Sibuya, M. and Shimizu, R. (1980a). *Classification of the Generalized Hypergeometric Family of Distributions*. (Monograph in preparation).

Sibuya, M. and Shimizu, R. (1980b). What are the generalized hypergeometric distributions? (To appear).

Simon, H. A. (1955). On a class of skew distribution functions. *Biometrika*, 42, 425-440.

Simon, H. A. (1960). Some further notes on a class of skew distribution functions. *Information and Control*, 3, 80-88.

Skibinsky, M. (1970). A characterization of hypergeometric distributions. *Journal of the American Statistical Association*, 65, 926-929.

Xekalaki, E. (1980). On an inventory model with a Yule demand distribution. Research Report No. EX01, Statistics and Operations Research Laboratory, Trinity College, Dublin.

Yule, G. W. (1924). A mathematical theory of evolution based on the conclusions of Dr. J. C. Willis, F.R.S. *Philosophical Transactions of the Royal Society of London, Series B*, 213, 21-87.

[*Received May* 1980. *Revised October* 1980]