

## **Factorial moment estimation for the bivariate generalized Waring distribution**

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In the context of accident theory, the bivariate generalized Waring distribution (Xekalaki, 1984) is known to offer the possibility of obtaining distinguishable estimates of the "contribution" of chance, risk exposure and proneness to an accident situation. In this paper an estimation procedure based on the first and second order factorial moments is discussed for fitting the distribution to data. Expressions for the asymptotic standard errors of the estimators of the distribution parameters as well as of the resulting estimators of the variance components that represent the roles of the above mentioned factors are given.

**Keywords and phrases:** Generalized Waring distribution; factorial moment method of estimation; asymptotic standard errors; variance components.

### 1. INTRODUCTION

Accidents, being a social problem, have been the object of a number of studies aiming at preventing their occurrence by enhancing our knowledge of the underlying mechanisms. Recently, Xekalaki (1984) introduced a bivariate discrete distribution, the bivariate generalized Waring distribution (BGWD), for the study of the accident experience of a population in two successive time periods in the context of an accident-proneness accident-liability model. The probability function (p.f.) of this distribution is given by

$$P(X=r, Y=\ell) = \frac{\rho^{(k+m)}}{(a+\rho)_{(k+m)}} \frac{a^{(r+\ell)} k^{(r)} m^{(\ell)}}{(a+k+\rho+m)_{(r+\ell)}} \frac{1}{r!} \frac{1}{\ell!} \quad (1.1)$$

$$r=0,1,2,\dots; \ell=0,1,2,\dots; a,k,m,\rho > 0$$

with  $h_{(s)}$  for  $\Gamma(h+s)/\Gamma(h)$ ,  $h>0$ ,  $s \in \mathbb{R}$ . Here  $X$  and  $Y$  denote the numbers of accidents in the first and second period respectively.

The accident model considered was based on the assumption that, apart from chance, non-random factors contribute to the happening of an accident and that these can be distinguished into factors related to the person's idiosyncrasy predisposition to accidents (accident proneness) and factors related to the person's exposure to external accident risk (accident liability). The BGWD then arises from a double Poisson distribution of accidents with probability-generating function  $\exp\{(\lambda_1|\nu)(s-1)+(\lambda_2|\nu)(t-1)\}$  for individuals with proneness  $\nu$  and liability  $\lambda_i|\nu$  for period  $i$  of observation ( $i=1,2$ ) when  $\lambda_1|\nu$  and  $\lambda_2|\nu$  have independent gamma distributions and  $\nu$  has a beta distribution of the second kind.

It was shown by Xekalaki (1984) that in an accident situation that can be described adequately by the BGWD the effects of the threetypes of factors can be "measured" separately. That was achieved by splitting the total variance of the observations into three additive components corresponding to the effects of the three types of factors i.e.

$$\sigma^2 = \sigma_\lambda^2 + \sigma_\nu^2 + \sigma_R^2 \quad (1.2)$$

where  $\sigma_\lambda^2$ ,  $\sigma_\nu^2$  and  $\sigma_R^2$  represent the liability, proneness and random components respectively. These can be expressed in terms of the parameters  $a, k, m$  and  $\rho$  thus

$$\sigma_\lambda^2 = \frac{a(k+m)(a+1)}{(\rho-1)(\rho-2)}, \quad \sigma_\nu^2 = \frac{a(k+m)^2(a+\rho-1)}{(\rho-1)^2(\rho-2)}, \quad \sigma_R^2 = \frac{a(k+m)}{\rho-1}. \quad (1.3)$$

Therefore, any method of estimation of  $a, k, m$  and  $\rho$  will lead to estimators for the above variance components and hence for the effects of the causing factors. This fact is of significant practical value as it throws some light onto the mechanisms that may cause individuals to have accidents and it can possibly lead to what is the ultimate objective: a way to reduce the incidence of accidents. This problem was originally tackled by Irwin (1968) who used the univariate version of the generalized Waring distribution (UGWD). However, he was led to indistinguishable estimates for the proneness and liability variance components.

To illustrate the bivariate approach, Xekalaki (1984) fitted the BGWD to two sets of accident data using the first and second order factorial moments of the distribution. She then used the resulting parameter estimates to estimate the three variance components and assess the effects of the three causing factors.

In this paper a greater insight will be given into the principle of the estimation procedure used by Xekalaki (1984). In particular, in section 3 explicit formulae for the distribution parameter estimators will be given as well as for their variance-covariance matrix in terms of the parameters of the distribution. Moreover, the asymptotic standard errors of the estimators of the variance components are obtained through expressions involving the parameters of the BGWD. Some background information is first given in section 2.

## 2. SOME PRELIMINARY REMARKS

The univariate version of the distribution described in the previous section (UGWD) has p.f. given by

$$P(X=r) = \frac{\rho^{(k)}}{(a+\rho)_{(k)}} \frac{a^{(r)} k^{(r)}}{(a+k+\rho)_{(r)}} \frac{1}{r!}, \quad r=0,1,2,\dots$$

For more information about the structure, properties and applications of this distribution see Xekalaki (1981,1983 a,b).

It was shown by Xekalaki (1984) that the marginal dis-

tributions of (1.1) as well as their convolution and the conditional distributions of one margin given the other are of the above form. Specifically, if  $(X, Y) \sim \text{BGWD}(a; k, m; \rho)$  then  $X \sim \text{UGWD}(a, k; \rho)$ ,  $Y \sim \text{UGWD}(a, m; \rho)$ ,  $X+Y \sim \text{UGWD}(a, k+m; \rho)$ ,  $X|Y=y \sim \text{UGWD}(a+y, k; \rho+m)$  and  $Y|X=x \sim \text{UGWD}(a+x, m; \rho+k)$ . Beyond the general interest that these properties have (bearing an analogy to those of the bivariate normal distribution in the continuous case), their practical importance should be noted. In an accident situation, for example, one would naturally require the marginal accident distributions and the distribution of accidents over the entire period to be of the same form. Moreover, the first three properties imply the possibility of inferring about the contribution of proneness, liability and chance in each of the two subperiods. This possibility stems from the fact that the marginal variances  $\sigma_X^2$  and  $\sigma_Y^2$  can be split into estimable components in a manner analogous to that exhibited by (1.2). Table 1 summarizes the estimating potential of the model.

TABLE 1

Estimators of the components of the variance of the generalized Waring distribution.

Component due to	Marginal variance of X	Marginal variance of Y	Variance of X+Y
Random factors	$\frac{\hat{a} \hat{k}}{\hat{\rho}-1}$	$\frac{\hat{a} \hat{m}}{\hat{\rho}-1}$	$\frac{\hat{a}(\hat{k}+\hat{m})}{\hat{\rho}-1}$
Proneness	$\frac{\hat{k}^2 \hat{a}(\hat{a}+\hat{\rho}-1)}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$	$\frac{\hat{m}^2 \hat{a}(\hat{a}+\hat{\rho}-1)}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$	$\frac{(\hat{k}+\hat{m})^2 \hat{a}(\hat{a}+\hat{\rho}-1)}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$
Liability	$\frac{\hat{a} \hat{k}(\hat{a}+1)}{(\hat{\rho}-1)(\hat{\rho}-2)}$	$\frac{\hat{a} \hat{m}(\hat{a}+1)}{(\hat{\rho}-1)(\hat{\rho}-2)}$	$\frac{\hat{a}(\hat{k}+\hat{m})(\hat{a}+1)}{(\hat{\rho}-1)(\hat{\rho}-2)}$
Total	$\frac{\hat{a} \hat{k}(\hat{\rho}+\hat{k}-1)(\hat{\rho}+\hat{a}-1)}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$	$\frac{\hat{a} \hat{m}(\hat{\rho}+\hat{m}-1)(\hat{\rho}+\hat{a}-1)}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$	$\frac{\hat{a}(\hat{k}+\hat{m})(\hat{\rho}+\hat{k}+\hat{m}-1)(\hat{\rho}+\hat{a}-1)}{(\hat{\rho}-1)^2(\hat{\rho}-2)}$

Here  $\hat{\theta}$  represents an estimator of a parameter  $\theta$ .

The factorial moments of the BGWD  $(a; k, m; \rho)$  are given by the formula

$$\mu_{(i,j)} = \frac{a_{(i+j)} k_{(i)} m_{(j)}}{(\rho-1)(\rho-2)\dots(\rho-i-j)}, \quad \begin{array}{l} i=0,1,2,\dots \\ j=0,1,2,\dots \\ i+j \geq \rho \end{array} \quad (2.1)$$

from which one can obtain for the moments of the BGWD

$$\begin{aligned} \mu'_{1,0} = \mu_X &= \frac{ak}{\rho-1}, \quad \mu'_{0,1} = \mu_Y = \frac{am}{\rho-1}, \quad \mu_{1,1} = \sigma_{XY} = \frac{akm(a+\rho-1)}{(\rho-1)^2(\rho-2)} \\ \mu_{2,0} = \sigma_X^2 &= \frac{ak(\rho+k-1)(\rho+a-1)}{(\rho-1)^2(\rho-2)}, \quad \mu_{0,2} = \sigma_Y^2 = \frac{am(\rho+m-1)(\rho+a-1)}{(\rho-1)^2(\rho-2)}. \end{aligned}$$

### 3. ASYMPTOTIC STANDARD ERRORS

As mentioned before, Xekalaki (1984) fitted the BGWD to accident data by a method that employs the first and second order factorial moments of the distribution. In particular, if  $X, Y$  denote the numbers of accidents during the first and second period respectively, the associated estimating equations are

$$\begin{aligned} \bar{X} &= \frac{\hat{a} \hat{k}}{\hat{\rho}-1} & \bar{Y} &= \frac{\hat{a} \hat{m}}{\hat{\rho}-1} \\ \bar{W} + \bar{Z} &= \frac{\hat{a}(\hat{a}+1) [\hat{k}(\hat{k}+1) + \hat{m}(\hat{m}+1)]}{(\hat{\rho}-1)(\hat{\rho}-2)} & \bar{T} &= \frac{\hat{a}(\hat{a}+1) \hat{k} \hat{m}}{(\hat{\rho}-1)(\hat{\rho}-2)} \quad (3.1) \end{aligned}$$

where

$$\bar{W} = \frac{1}{n} \sum_{i,j} j(j-1) f_{ij}, \quad \bar{Z} = \frac{1}{n} \sum_{i,j} i(i-1) f_{ij},$$

$\bar{T} = \frac{1}{n} \sum_{i,j} f_{ij}$ ,  $n = \sum_{i,j} f_{ij}$  and  $f_{ij}$  is the observed

joint frequency. As a solution, we get

$$\hat{k} = \frac{\bar{X}\bar{T}(\bar{X}+\bar{Y})}{\bar{X}\bar{Y}(\bar{W}+\bar{Z}) - \bar{T}(\bar{X}^2+\bar{Y}^2)}, \quad \hat{m} = \frac{\hat{k}\bar{Y}}{\bar{X}} \quad (3.2)$$

$$\hat{a} = \frac{\hat{k}\bar{X}\bar{Y}+\bar{T}\bar{X}}{\hat{k}(\bar{T} - \bar{X}\bar{Y})}, \quad \hat{\rho} = \frac{\hat{a}\hat{k}+\bar{X}}{\bar{X}}.$$

The asymptotic variance-covariance matrix  $V$  of these parameter estimators can be obtained as follows:

Let  $\underline{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$  denote the parameter vector  $(a, k, m, \rho)$ ,  $\hat{\underline{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4)$  denote the vector  $(\hat{a}, \hat{k}, \hat{m}, \hat{\rho})$  and let  $\underline{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4)$  and  $\underline{t} = (t_1, t_2, t_3, t_4)$  denote the vectors  $(\mu_X, \mu_Y, \mu(2,0) + \mu(0,2), \mu(1,1))$  and

$(\bar{X}, \bar{Y}, \bar{W}+\bar{Z}, \bar{T})$  respectively.

(Here  $\mu(r, \ell) = E(X^{(r)} Y^{(\ell)})$  where  $X^{(r)} = X(X-1)\dots(X-r+1)$ ,  $X^{(0)} = 1$ ).

The parameter estimators  $\hat{\theta}_i$  are functions of the sample factorial moments  $t_i$ , say

$$\hat{\theta}_i = \phi_i(t_1, t_2, t_3, t_4), \quad i=1,2,3,4.$$

Then, from the general theory we have

$$V = J S J', \quad (3.3)$$

where

$$J = (J_{ij}) = \left( \frac{\partial \phi_i}{\partial \hat{\theta}_j} \right)_{\hat{\theta}=\underline{\theta}}^{-1} = \left( \frac{\partial \phi_i}{\partial t_j} \right)_{\underline{t}=\underline{\tau}}, \quad \begin{matrix} i = 1,2,3,4 \\ j = 1,2,3,4 \end{matrix} \quad (3.4)$$

and

$$S = (S_{ij}) = (\text{Cov}(t_i, t_j)), \quad \begin{matrix} i = 1,2,3,4 \\ j = 1,2,3,4 \end{matrix} \quad (3.5)$$

where  $J'$  is the transpose of  $J$ . Substituting for the partial derivatives in (3.4) from (3.1) (or (3.2)) and replacing  $\hat{\theta}_i$  by  $\theta_i$  (or  $t_i$  by  $\tau_i$ ) we obtain

$$J = (\rho-1)^{-1} \times \left[ \begin{array}{cccc} k & a & 0 & -\frac{ak}{\rho-1} \\ m & 0 & a & -\frac{am}{\rho-1} \\ \frac{(2a+1)(k^2+m^2+k+m)}{\rho-2} & \frac{a(a+1)(2k+1)}{\rho-2} & \frac{a(a+1)(2m+1)}{\rho-2} & -\frac{a(a+1)(2\rho-3)(k^2+m^2+k+m)}{(\rho-1)(\rho-2)^2} \\ \frac{(2a+1)km}{\rho-2} & \frac{ma(a+1)}{\rho-2} & \frac{ka(a+1)}{\rho-2} & -\frac{akm(a+1)(2\rho-3)}{(\rho-1)(\rho-2)^2} \end{array} \right]^{-1}$$

Inverting the matrix on the right hand side of the above formula yields after much algebra the following expression for  $J$ :

$$J = (\rho-1) (k+m)^{-1} \times \left[ \begin{array}{cccc} \frac{(a+1)[(k+m)(m-k+\rho-1)-k]}{k(a+\rho-1)} & \frac{(a+1)[(k+m)(k-m+\rho-1)-m]}{m(a+\rho-1)} & \frac{\rho-2}{a+\rho-1} & \frac{(\rho-2)[(\rho-1)(k+m)+k^2+m^2]}{km(a+\rho-1)} \\ \frac{k^2-m^2+k}{a} & \frac{k(m^2-k^2-k)}{am} & -\frac{k(\rho-2)}{a(a+1)} & \frac{(\rho-2)(k^2+m^2+k+m)}{ma(a+1)} \\ \frac{m(k^2-m^2-m)}{ak} & \frac{m^2-k^2+m}{a} & -\frac{m(\rho-2)}{a(a+1)} & \frac{(\rho-2)(k^2+m^2+k+m)}{ka(a+1)} \\ \frac{(\rho-1)(\rho-2)[k+(k+m)(k-m+a)]}{ak(a+\rho-1)} & \frac{(\rho-1)(\rho-2)[m+(k+m)(m-k+a)]}{am(a+\rho-1)} & -\frac{(\rho-1)(\rho-2)^2}{a(a+1)(a+\rho-1)} & \frac{(\rho-1)(\rho-2)^2[k^2+m^2-a(k+m)]}{akm(a+1)(a+\rho-1)} \end{array} \right]$$

Also the matrix  $S$  is given by

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{\bar{X}}^2 \\ \sigma_{\bar{YX}} \sigma_{\bar{Y}}^2 \\ \sigma_{\bar{WX}} \sigma_{\bar{WY}} \sigma_{\bar{W}}^2 \\ \sigma_{\bar{ZX}} \sigma_{\bar{ZY}} \sigma_{\bar{ZW}} \sigma_{\bar{Z}}^2 \\ \sigma_{\bar{TX}} \sigma_{\bar{TY}} \sigma_{\bar{TW}} \sigma_{\bar{TZ}} \sigma_{\bar{T}}^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.6)$$

If we let  $m_{r,s}^-$  denote the sample statistic  $\sum_{i,j} i^r j^s f_{ij}/n$ , the elements of the symmetrical matrix in (3.6) can be derived by observing that

$$\sigma_{\bar{X}}^2 = \frac{1}{n} \sigma_X^2, \quad \sigma_{\bar{Y}}^2 = \frac{1}{n} \sigma_Y^2$$

$$\sigma_{\bar{W}}^2 = \text{Var}(m_{0,2}^-) + \sigma_{\bar{Y}}^2 - 2\text{Cov}(m_{0,2}^-, m_{0,1}^-)$$

$$\sigma_{\bar{Z}}^2 = \text{Var}(m_{2,0}^-) + \sigma_{\bar{X}}^2 - 2\text{Cov}(m_{1,0}^-)$$

$$\sigma_{\bar{T}}^2 = \text{Var}(m_{1,1}^-), \quad \sigma_{\bar{XY}} = \frac{1}{n} \sigma_{XY}$$

$$\sigma_{\bar{WX}} = \text{Cov}(m_{1,0}^-, m_{0,2}^-) - \sigma_{\bar{XY}}, \quad \sigma_{\bar{ZX}} = \text{Cov}(m_{1,0}^-, m_{2,0}^-) - \sigma_{\bar{X}}^2$$

$$\sigma_{\bar{TX}} = \text{Cov}(m_{1,0}^-, m_{1,1}^-), \quad \sigma_{\bar{WY}} = \text{Cov}(m_{0,1}^-, m_{0,2}^-) - \sigma_{\bar{Y}}^2$$

$$\sigma_{\bar{ZY}} = \text{Cov}(m_{0,1}^-, m_{2,0}^-) - \sigma_{\bar{XY}}, \quad \sigma_{\bar{TY}} = \text{Cov}(m_{0,1}^-, m_{1,1}^-)$$

$$\sigma_{\bar{ZW}} = \text{Cov}(m_{2,0}^-, m_{0,2}^-) - \text{Cov}(m_{2,0}^-, m_{0,1}^-) - \text{Cov}(m_{1,0}^-, m_{0,2}^-) \\ + \sigma_{\bar{XY}}$$

$$\sigma_{\bar{TW}} = \text{Cov}(m_{1,1}^-, m_{0,2}^-) - \text{Cov}(m_{1,1}^-, m_{0,1}^-)$$

$$\sigma_{\bar{TZ}} = \text{Cov}(m_{1,1}^-, m_{2,0}^-) - \text{Cov}(m_{1,1}^-, m_{1,0}^-)$$



where (see Kendall and Stuart (1977, p. 250))

$$\text{Cov}(m_{r,s}^-, m_{u,v}^-) = \frac{1}{n} (\mu_{r+u,s+v}^- - \mu_{r,s}^- \mu_{u,v}^-),$$

$$\mu_{r,s}^- = E(X^r Y^s), \quad r, s = 0, 1, 2, \dots$$

Therefore the elements of S are given by

$$S_{11} = \frac{1}{n} \sigma_X^2, \quad S_{12} = S_{21} = \frac{1}{n} \sigma_{XY}$$

$$S_{13} = S_{31} = \frac{1}{n} \{ \mu_{1,2}^- - \mu_X (\mu_{0,2}^- + \mu_{2,0}^-) + \mu_{3,0}^- - \sigma_{XY} - \sigma_X^2 \}$$

$$S_{14} = S_{41} = \frac{1}{n} \{ \mu_{2,1}^- - \mu_X \mu_{1,1}^- \}, \quad S_{22} = \frac{1}{n} \sigma_Y^2$$

$$S_{23} = S_{32} = \frac{1}{n} \{ \mu_{2,1}^- - \mu_Y (\mu_{0,2}^- + \mu_{2,0}^-) + \mu_{0,3}^- - \sigma_{XY} - \sigma_Y^2 \}$$

$$S_{24} = S_{42} = \frac{1}{n} \{ \mu_{1,2}^- - \mu_Y \mu_{1,1}^- \}$$

$$S_{33} = \frac{1}{n} \{ \mu_{4,0}^- + \mu_{0,4}^- - (\mu_{2,0}^- + \mu_{0,2}^-)^2 + 2\mu_{2,2}^-$$

$$- 4(\mu_{2,1}^- + \mu_{1,2}^- - \mu_{2,0}^- \mu_Y - \mu_{0,2}^- \mu_X) + \sigma_X^2 + \sigma_Y^2 + 2S_{XY} \}$$

$$S_{34} = S_{43} = \frac{1}{n} \{ \mu_{3,1}^- + \mu_{1,3}^- - \mu_{1,1}^- (\mu_{2,0}^- + \mu_{0,2}^- - \mu_{0,2}^- \mu_X - \mu_{2,0}^- \mu_Y) - \mu_{1,2}^-$$

$$- \mu_{2,1}^- \}$$

$$S_{44} = \frac{1}{n} \{ \mu_{2,2}^- - \mu_{1,1}^{-2} \}.$$

Expressions for the central moments in terms of the parameters  $a, k, m$  and  $\rho$  can be obtained using (2.1) through the formula

$$\mu_{r,\ell}^{\sim} = \sum_{i=0}^r \sum_{j=0}^{\ell} S_r^i S_{\ell}^j \mu_{(i,j)}, \quad (3.7)$$

where  $S_m^n$  is the Stirling number of the second kind. (See Jordan (1965), p.170). Thus finally, after some simplification one obtains for the elements of S the following expressions :

$$S_{11} = \frac{ak(\rho+k-1)(\rho+a-1)}{n(\rho-1)^2(\rho-2)}, \quad S_{12} = S_{21} = \frac{akm(\rho+a-1)}{n(\rho-1)^2(\rho-2)}$$

$$S_{13} = S_{31} = \frac{ak}{n(\rho-1)(\rho-2)} \times$$

$$\left\{ \frac{(a+\rho-1)(m-k-\rho+1) - a[(k+m)(a+\rho-1) + (k^2+m^2)(a+1)]}{\rho-1} \right. \\ \left. + \frac{(a+1)[m(m+1)(a+2) + (k+1)(3\rho+ak+2a+2k-5)]}{\rho-3} + m(a+1) + \rho - 2 \right\}$$

$$S_{14} = S_{41} = \frac{a(a+1)km[(\rho-1)(\rho-3) + (a+2)(k+1)(\rho-1) - ak(\rho-3)]}{n(\rho-1)^2(\rho-2)(\rho-3)}$$

$$S_{22} = \frac{am(\rho+m-1)(\rho+a-1)}{n(\rho-1)^2(\rho-2)}, \quad S_{23} = S_{32} = \frac{am}{n(\rho-1)(\rho-2)} \times$$

$$\left\{ \frac{(a+\rho-1)(k-m-\rho+1) - a[(k+m)(a+\rho-1) + (k^2+m^2)(a+1)]}{\rho-1} \right. \\ \left. + \frac{(a+1)[k(k+1)(a+2) + (m+1)(3\rho+am+2a+2m-5)]}{\rho-3} + k(a+1) + \rho - 2 \right\}$$

$$S_{24} = S_{42} = \frac{a(a+1)km[(\rho-1)(\rho-3) + (a+2)(m+1)(\rho-1) - am(\rho-3)]}{n(\rho-1)^2(\rho-2)(\rho-3)}$$

$$S_{33} = \frac{a}{n(\rho-1)} \left\{ -k-m - \frac{a(a+1)^2(k^2+m^2+k+m)^2}{(\rho-1)(\rho-2)^2} + \right.$$

$$\begin{aligned}
& + \frac{(k+m) [(a+1)(\rho-1) + (\rho+k+m-1)(\rho-a+1)] + (k^2+m^2)(2a\rho-3a+\rho-1)}{(\rho-1)(\rho-2)} \\
& + \frac{(a+1)(a+2) [k(k+1)(k+2)(ak+3a+3k+4\rho-7)]}{(\rho-2)(\rho-3)(\rho-4)} \\
& + \frac{(a+1)(a+2) [m(m+1)(m+2)(am+3a+3m+4\rho-7) + 2(a+3)km(k+1)(m+1)]}{(\rho-2)(\rho-3)(\rho-4)} \\
S_{34} = S_{43} = & \frac{a(a+1)km}{n(\rho-1)(\rho-2)} \left\{ - \frac{a(a+1)(m^2+k^2+m+k)}{(\rho-1)(\rho-2)} \right. \\
& \left. + \frac{(a+2) [(a+3)(m^2+k^2+3m+3k+4) + 2(\rho-4)(k+m+2)]}{(\rho-3)(\rho-4)} \right\} \\
S_{44} = & \frac{a(a+1)km}{n(\rho-1)(\rho-2)} \left\{ 1 - \frac{a(a+1)km}{(\rho-1)(\rho-2)} \right. \\
& \left. + \frac{(a+2) [(a+3)(k+1)(m+1) + (\rho-4)(k+m+2)]}{(\rho-3)(\rho-4)} \right\} .
\end{aligned}$$

Hence, for a given practical situation the estimators of the variance components as given by Table 1 will have asymptotic variances obtained by

$$V(\hat{\sigma}_{ij}^2) = I_{ij}^{-1} V I_{ij} \quad (3.8)$$

with  $i$  for component type, i.e., random ( $i=1$ ), proneness ( $i=2$ ), liability ( $i=3$ ) and  $j$  for time period, i.e., first ( $j=1$ ), second ( $j=2$ ) overall ( $j=3$ ). Here

$$I_{ij} = \left( \frac{\partial \hat{\sigma}_{ij}^2}{\partial \hat{a}}, \frac{\partial \hat{\sigma}_{ij}^2}{\partial \hat{k}}, \frac{\partial \hat{\sigma}_{ij}^2}{\partial \hat{m}}, \frac{\partial \hat{\sigma}_{ij}^2}{\partial \hat{\rho}} \right)_{\hat{\theta}=\theta} \quad (3.9)$$

$i = 1, 2, 3, \quad j = 1, 2, 3$  which substituting for the appropriate partial derivatives, reduce to

$$I_{11} = \frac{1}{\rho-1} \left( k, a, 0, -\frac{ak}{\rho-1} \right)$$

$$I_{12} = \frac{1}{\rho-1} \left( m, 0, a, -\frac{am}{\rho-1} \right)$$

$$I_{13} = \frac{1}{\rho-1} \left( k+m, a, a, -\frac{a(k+m)}{\rho-1} \right)$$

$$I_{21} = \frac{k}{(\rho-1)^2(\rho-2)} \left( k(2a+\rho-1), 2a(a+\rho-1), 0, \frac{ak[-2\rho^2+(5-3a)\rho+5a-3]}{(\rho-1)(\rho-2)} \right)$$

$$I_{22} = \frac{m}{(\rho-1)^2(\rho-2)} \left( m(2a+\rho-1), 0, 2a(a+\rho-1), \frac{am[-2\rho^2+(5-3a)\rho+5a-3]}{(\rho-1)(\rho-2)} \right)$$

$$I_{23} = \frac{k+m}{(\rho-1)^2(\rho-2)} \left( (k+m)(2a+\rho-1), 2a(a+\rho-1), 2a(a+\rho-1), \frac{a(k+m)[-2\rho^2+(5-3a)\rho+5a-3]}{(\rho-1)(\rho-2)} \right)$$

$$I_{31} = \frac{1}{(\rho-1)(\rho-2)} \left( k(2a+1), a(a+1), 0, -\frac{ak(a+1)(2\rho-3)}{(\rho-1)(\rho-2)} \right)$$

$$I_{32} = \frac{1}{(\rho-1)(\rho-2)} \left( m(2a+1), 0, a(a+1), -\frac{am(a+1)(2\rho-3)}{(\rho-1)(\rho-2)} \right)$$

$$I_{33} = \frac{1}{(\rho-1)(\rho-2)} \left( (k+m)(2a+1), a(a+1), a(a+1), -\frac{a(k+m)(a+1)(2\rho-3)}{(\rho-1)(\rho-2)} \right).$$

#### 4. AN EXAMPLE

To illustrate the described procedure for the evaluation of the standard errors involved in the estimation of the parameters of the BGWD and of the variance components, consider the motor-vehicle accident data of Table 2 (U.S. Bureau of Public Roads, 1938) to which Xekalaki (1984) fitted the BGWD.

TABLE 2(\*)  
Accidents to Connecticut General Drivers  
 1931 - 1933

X \ Y	0	1	2	3	4	MARG X	
1934 - 1936	0	23881 23881.17	2117 2144.91	242 213.43	17 23.26	2 2.75	26259 26265.52
	1	2386 2374.65	419 422.20	57 62.36	9 8.97	3 1.31	2874 2869.49
	2	275 258.90	64 68.33	12 13.32	5 2.37	1 0.41	357 343.33
	3	22 30.65	5 10.68	2 2.57	2 0.54	0 0.11	31 44.56
	4	5 3.91	4 1.68	0 0.48	1 0.12	0 0.03	10 6.22
MARG Y	26569 26549.29	2609 2647.80	313 292.17	34 35.26	6 4.61	29531 29529.13	

$\chi^2 = 17.9978$  with 10 degrees of freedom

$P(\chi_{10}^2 \geq 17.9978) = .056$

(\*) Broken lines indicate the grouping adopted in the application of the Pearson chi-square test.

The estimates of the distribution parameters were obtained from (3.2). In particular,  $\hat{a} = 0.9992$ ,  $\hat{k} = 9.2774$ ,  $\hat{m} = 8.3798$  and  $\hat{\rho} = 74.5709$ . Their asymptotic variance covariance matrix estimated by computing and inverting the matrix  $\left( \frac{\partial \phi_i^{-1}}{\partial \theta_j} \right)$  comes out to be

	$\hat{a}$	$\hat{k}$	$\hat{m}$	$\hat{\rho}$
$\hat{a}$	0.0088			
$\hat{k}$	-0.4315	68.5270		
$\hat{m}$	-0.3899	61.8767	55.9126	
$\hat{\rho}$	-2.7734	511.4730	461.9780	3852.16

yielding 0.0938, 8.2781, 7.4775 and 62.0658 for the standard errors of  $\hat{a}$ ,  $\hat{k}$ ,  $\hat{m}$ , and  $\hat{\rho}$  respectively. The corresponding standard errors of the estimates of the components of the variance obtained through (3.9) and (3.10) are indicated in the following table (Table 3).

TABLE 3  
Estimates of the components of the variance

Component	Time period		
	1931-1933	1934-1936	1931-1936
Random	0.1260 ± 0.0022	0.1138 ± 0.0021	0.2398 ± 0.0032
Proneness	0.0163 ± 0.0015	0.0133 ± 0.0012	0.0591 ± 0.0053
Liability	0.0035 ± 0.0031	0.0031 ± 0.0027	0.0066 ± 0.0058

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