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MODELS LEADING TO THE
BIVARIATE GENERALIZED WARING DISTRIBUTION

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ABSTRACT. This paper is concerned with the study of models that generate the bivariate generalized Waring distribution. Urn, mixing, conditionality, STER and exceedance models are considered as well as models based on the structural properties of the distribution. The bivariate generalized Waring distribution is shown to belong to the Pearson's system of bivariate discrete distributions and its relationship with bivariate continuous Pearson distributions is examined. Limiting forms of both discrete and continuous type are derived.

Introduction.

The bivariate generalized Waring distribution with parameters $a > 0$, $k > 0$, $m > 0$ and $\rho > 0$ (BGWD$(a;k,m;\rho)$) has been studied by Xekalaki (1977,1984). It is the probability distribution of a random vector (X,Y) of nonnegative, integer-valued components with probability function (p.f.) given by

$$(1.1) \qquad p_{x,y} \equiv P(X=x,Y=y) = \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} \frac{a_{(x+y)}k_{(x)}m_{(y)}}{(a+k+m+\rho)_{(x+y)}} \frac{1}{x!}\frac{1}{y!}$$

where $\alpha_{(\beta)} = \Gamma(\alpha+\beta)/\Gamma(\alpha)$, $\alpha > 0$ $\beta \in R$. The p.f. in (1.1) was obtained as the general term of a bivariate series of ascending factorials defining a two-dimensional extension of Waring's series expansion of a function of the form $(x-a)^{-1}_{(k+m)}$, $x > a > 0$; $k,m > 0$. The probability generating function (p.g.f.) is

$$(1.2) \qquad G(s,t) = \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} F_1(a;k,m;a+k+m+\rho;s,t), \quad (s,t)\in[-1,1]\times[-1,1]$$

where F_1 is the Appell hypergeometric function of the first type defined by

$$F_1(a;b,b';c;z,w) = \sum_{r=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{a_{(r+\ell)} b_{(r)} b'_{(\ell)}}{c_{(r+\ell)}} \frac{z^r}{r!} \frac{w^\ell}{\ell!}$$

for $a,b,b',c-a-b-b' > 0$ and $(z,w) \in [-1,1] \times [-1,1]$. The random variables (r.v.'s) X, Y, $X \mid (Y=y)$, $Y \mid (X=x)$, $X+Y$ follow univariate generalized Waring distributions (UGWD) with appropriate parameters. (The UGWD has been defined by Irwin (1963,1968,1975) and studied by Xekalaki (1981, 1983a,b).)

The derivation of the BGWD was motivated by a problem in accident theory to which the univariate version did not provide an entirely satisfactory solution. This problem was concerned with providing separate estimates for the variance components corresponding to random, psychological and external factors that may have contributed to a given accident situation. The effects of the last two kinds of factors as measured by the variance components due to them were confounded. Indeed, as it was shown by Xekalaki (1984), presenting the data in a bivariate form with reference to two consecutive and nonoverlapping time periods and fitting the BGWD enables one to "measure" these effects in terms of separate estimates of the corresponding variance components. Beyond this result and its important implications, the fact that the marginal and conditional distributions as well as the convolution are of the UGWD type suggests that the BGWD has potential applications in as large a number of fields as the UGWD has. (For a detailed account of the applications of the UGWD, see Xekalaki (1981) and the references therein.) Naturally then, studying possible models that give rise to the BGWD may be of interest. The following sections of this paper deal with such models. In particular, urn models, mixing and conditionality models are considered. Also some derivations of the BGWD are given in terms of STER and exceedance models as well as a genesis scheme based on a property of the tail probabilities of this distribution. Further, the BGWD is demonstrated to be a member of the Pearson system of discrete distributions and its relationship to the Pearson system of continuous distributions is studied. Finally, it is shown that the BGWD tends to the double negative binomial distribution and the double Poisson distribution. It is also shown that by appropriately choosing the scale, it can have a bivariate beta II distribution (Dirichlet type II or Pearson type II$\alpha\beta$) or a bivariate gamma distribution with independent components as limiting cases.

1. *Urn Models.*

1.1. *Sampling from an urn with three types of balls.*

Consider an urn containing 'a' red, 'b' black and 'c' white balls. One ball is drawn at random and replaced together with one additional ball of the same colour before the next ball is drawn. The numbers X of red balls and Y of black balls drawn before the ℓ^{th} white have the BGWD $(\ell;a,b;c)$ as their joint distribution, i.e.,

$$P[X=x,Y=y] = \frac{c^{(b+a)}}{(c+\ell)_{(b+a)}} \; \frac{\ell^{(x+y)}a^{(x)}b^{(y)}}{(a+b+\ell+c)_{(x+y)}} \; \frac{1}{x!} \; \frac{1}{y!} \; .$$

Polya's bivariate inverse urn scheme where each sampled ball is replaced by h+1 balls of the same colour before the next ball is drawn, is a more general case, and the bivariate inverse Polya with parameters a,b_1,b_2,h and ℓ is the same as the BGWD $\left(\ell;\frac{b_1}{h},\frac{b_2}{h};\frac{a}{h}\right)$.

1.2 *Sampling from an urn with two types of balls.*

The urn contains 'a' white and 'b' black balls. One ball is drawn at random and replaced along with 1 additional ball of the same colour before the next ball is drawn. The numbers X of black balls drawn before the first k white balls and Y of black balls drawn before the next m white balls (and after the first k) have a joint probability distribution defined by the p.f.

$$P[X=x,Y=y] = \frac{a^{(k+m)}}{(a+b)_{(k+m)}} \; \frac{b^{(x+y)}k^{(x)}m^{(y)}}{(a+b+k+m)_{(x+y)}} \; \frac{1}{x!} \; \frac{1}{y!}$$

To show this let B_i and W_i denote the events {black ball in the i^{th} drawing} and {white ball in the j^{th} drawing} respectively. Then,

$$P(X=x,Y=y) = P(B_1,B_2,\ldots,B_x,W_{x+1},W_{x+2},\ldots,W_{x+k},B_{x+k+1},B_{x+k+2},$$
$$\ldots,B_{x+k+y},W_{x+k+y+1},W_{x+k+y+2},\ldots,W_{x+k+y+m})$$
$$+ P(B_1,W_2,B_3,B_4,\ldots,B_{x+1},W_{x+2},W_{x+3},\ldots,W_{x+k+1},$$
$$B_{x+k+2},B_{x+k+3},\ldots,B_{x+k+y+1},W_{x+k+y+2},$$
$$W_{x+k+y+3},\ldots W_{x+y+k+m}) + \ldots$$
$$= \frac{b}{a+b}\frac{b+1}{a+b+1}\cdots\frac{b+x-1}{a+b+x-1}\frac{a}{a+b+x}\frac{a+1}{a+b+x+1}\cdots\frac{a+k-1}{a+b+x+k-1}\times$$
$$\frac{b+x}{a+b+x+k}\frac{b+x+1}{a+b+x+k+1}\cdots\frac{b+x+y-1}{a+b+x+y+k-1}\times$$
$$\frac{a+k}{a+b+x+y+k}\frac{a+k+1}{a+b+x+y+k+1}\cdots\frac{a+k+m-1}{a+b+x+y+k+m-1}+$$
$$+\frac{b}{a+b}\frac{a}{a+b+1}\frac{b+1}{a+b+2}\frac{b+2}{a+b+3}\cdots\frac{b+x-1}{a+b+x}\frac{a+1}{a+b+x+1}\frac{a+2}{a+b+x+2}\times$$
$$\cdots\frac{a+k-1}{a+b+x+k-1}\frac{b+x}{a+b+x+k}\frac{b+x+1}{a+b+x+k+1}\times$$
$$\cdots\frac{b+x+y-1}{a+b+x+k+y-1}\frac{a+k}{a+b+x+k+y}\frac{a+k+1}{a+b+x+y+k+1}\times$$
$$\cdots\frac{a+k+m-1}{a+b+x+y+k+m-1}+\ldots$$
$$= \frac{b^{(x+y)}a^{(k+m)}}{(a+b)^{(x+y+k+m)}} + \frac{b^{(x+y)}a^{(k+m)}}{(a+b)^{(x+y+k+m)}} + \ldots$$
$$= \binom{x+k-1}{x}\binom{y+m-1}{y}\frac{b^{(x+y)}a^{(k+m)}}{(a+b)^{(x+y+k+m)}}$$
$$\sim \text{BGWD}\,(b;k,m;a).$$

2. *Mixing Models.*

1.2. *Beta II mixtures of bivariate negative binomial distributions.*

Consider two r.v.'s X and Y whose joint distribution is the bivariate negative binomial with p.g.f.

$$g(s,t) = (1+Q_1(1-s)+Q_2(1-t))^{-a},\quad a,Q_1,Q_2 > 0\,.$$

Let (Q_1,Q_2) be a Beta II vector with probability density function (p.d.f.)

$$f(Q_1,Q_2) = \frac{\Gamma(k+m+\rho)}{\Gamma(k)\Gamma(m)\Gamma(\rho)} Q_1^{k-1} Q_2^{m-1}(1+Q_1+Q_2)^{-(k+m+\rho)}, \quad Q_1,Q_2,k,m,\rho > 0.$$

Then the distribution of (X,Y) has p.g.f.

$$G(s,t) = \frac{\Gamma(k+m+\rho)}{\Gamma(k)\Gamma(m)\Gamma(\rho)} \int_0^\infty \int_0^\infty Q_1^{k-1} Q_2^{m-1}(1+Q_1+Q_2)^{-(k+m+\rho)}$$

$$\times (1+Q_1(1-s)+Q_2(1-t))^{-a}dQ_1dQ_2,$$

i.e.,

$$p_{x,y} = \frac{\Gamma(k+m+\rho)\Gamma(a+x+y)}{\Gamma(k)\Gamma(m)\Gamma(\rho)\Gamma(a)x!y!} \int_0^\infty \int_0^\infty Q_1^{k+x-1} Q_2^{m+y-1}$$

$$\times (1+Q_1+Q_2)^{-(a+k+m+\rho+x+y)}dQ_1dQ_2$$

$$= \frac{\Gamma(k+m+\rho)\Gamma(a+x+y)\Gamma(k+x)\Gamma(m+y)\Gamma(a+\rho)}{\Gamma(k)\Gamma(m)\Gamma(\rho)\Gamma(a)\Gamma(a+k+m+\rho+x+y)x!y!}$$

$$\sim \text{BGWD } (a;k,m;\rho).$$

2.2. Beta I mixtures of bivariate negative binomial distributions.

By applying the simple transformation

$$q_i = \frac{Q_i}{1+Q_1+Q_2}, \quad i = 1,2$$

to the above-mentioned mixing process we obtain the BGWD as a mixture on (q_1,q_2) of the bivariate negative binomial distribution with p.g.f.

$$g(s,t) = (1-q_1-q_2)^a(1-q_1s-q_2t)^{-a}, \quad a,q_1,q_2 > 0, \; q_1+q_2 < 1,$$

if q_1,q_2 have a bivariate beta I (a Dirichlet type I) as joint distribution with p.d.f.

$$f(q_1,q_2) = \frac{\Gamma(k+m+\rho)}{\Gamma(k)\Gamma(m)\Gamma(\rho)} q_1^{k-1} q_2^{m-1}(1-q_1-q_2)^{\rho-1}$$

where $k > 0$, $m > 0$, $\rho > 0$, $0 < q_i < 1$, $i = 1,2$, i.e.,

$$G(s,t) = \frac{\Gamma(\rho+k+m)}{\Gamma(\rho)\Gamma(k)\Gamma(m)} \iint\limits_{\substack{q_1+q_2\le 1\\ q_1,q_2\ge 0}} q_1^{k-1} q_2^{m-1} (1-q_1-q_2)^{\rho+a-1} \quad (1-q_1 s-q_2 t)^{-a} dq_1 dq_2$$

$\sim$ BGWD $(a;k,m;\rho)$.

2.3. Beta II mixtures of double negative binomial distributions.

Consider (X,Y) to be a random vector whose distribution is the double negative binomial with p.g.f.

$$g(s,t) = (1+Q(1-s))^{-k}(1+Q(1-t))^{-m}, \; Q > 0, \; k,m > 0.$$

Assume that Q varies and that its distribution is the Beta of Type II (Pearson Type VI) with p.d.f.

$$f(Q) = \frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)} Q^{a-1}(1+Q)^{-(a+\rho)}, \; Q > 0, \; a > 0, \; \rho > 0.$$

Then the distribution of the random vector (X,Y) has p.g.f. given by

$$G(s,t) = \frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)} \int_0^\infty Q^{a-1}(1+Q)^{-(a+\rho)}(1+Q(1-s))^{-k}(1+Q(1-t))^{-m}dQ,$$

i.e.,

$$p_{x,y} = \frac{\Gamma(a+\rho)\Gamma(k+x)\Gamma(m+y)}{\Gamma(a)\Gamma(\rho)\Gamma(k)\Gamma(m)x!y!} \int_0^\infty Q^{a+x+y-1}(1+Q)^{-(a+\rho+k+m+x+y)}dQ$$

$$= \frac{\Gamma(a+\rho)\Gamma(k+x)\Gamma(m+y)\Gamma(a+x+y)\Gamma(\rho+k+m)}{\Gamma(a)\Gamma(\rho)\Gamma(k)\Gamma(m)\Gamma(a+k+\rho+m+x+y)x!y!}$$

$\sim$ BGWD $(a;k,m;\rho)$.

This model formed the theoretical basis for the derivation of the BGWD as an accident distribution by Xekalaki (1984).

2.4. Beta I mixtures of double negative binomial distributions.

The transformation

$$q = \frac{Q}{1+Q}$$

applied to the above described mixing process implies the derivation of the BGWD $(a;k,m;\rho)$ as the mixture on q of the double negative binomial distribution with p.g.f. given by

$$g(s,t) = (1-q)^{k+m}(1-qs)^{-k}(1-qt)^{-m}, \; 0 < q < 1, \; k > 0, \; m > 0,$$

if q is a r.v. having the Beta distribution of the first kind (Pearson's type I) with p.d.f.

$$h(q) = \frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)} q^{a-1}(1-q)^{\rho-1}, \; a > 0, \; \rho > 0, \; 0 < q < 1.$$

Then the p.g.f. of the random vector (X,Y) is

$$G(s,t) = \frac{\Gamma(a+\rho)}{\Gamma(a)\Gamma(\rho)} \int_0^1 q^{a-1}(1-q)^{k+m+\rho-1}(1-qs)^{-k}(1-qt)^{-m}dq$$

$$\sim \text{BGWD } (a;k,m;\rho).$$

2.5. Mixtures of Poisson and generalized Poisson distributions.

Bivariate negative binomial distributions result as gamma mixtures of the Poisson distribution or as Poissons generalized by logarithmic series distributions. It follows then that the BGWD can also arise as a mixture of such distributions as indicated by the models given below (see also Xekalaki (1977)). Note that mixing is denoted by $\wedge$ while generalizing is denoted by $\vee$.

$$\text{double Poisson } (\lambda_1,\lambda_2) \underset{\lambda_1,\lambda_2}{\wedge} \text{ double gamma } (k,m;b^{-1},b^{-1}) \underset{b}{\wedge} \text{ Beta II } (a;\rho)$$

$$\sim \text{BGWD } (a;k,m;\rho)$$

$$\text{double Poisson } (\lambda b_1,\lambda b_2) \underset{\lambda}{\wedge} \text{ gamma } (a;b_1^{-1} \; b_2^{-1}) \underset{b_1,b_2}{\wedge} \text{ biv. Beta II } (k,m;\rho)$$

$$\sim \text{BGWD } (a;k,m;\rho)$$

$$\text{double Poisson } (\lambda_1,\lambda_2) \underset{\lambda_1,\lambda_2}{\wedge} \text{ double gamma } (k,m;b^{-1},b^{-1}) \underset{\frac{b}{b+1}}{\wedge} \text{ Beta I } (a;\rho)$$

$$\sim \text{BGWD } (a;k,m;\rho)$$

$$\text{double Poisson }(\lambda b_1,\lambda b_2) \wedge \text{gamma }(a;b_1^{-1}\ b_2^{-1}) \underset{c_1,c_2}{\wedge} \text{biv. Beta I }(k,m;\rho)$$

$$\sim \text{BGWD }(a;k,m;\rho)$$

where $c_i = b_i/(1+b_1+b_2)$, $i = 1,2$.

$$\text{double Poisson }(-\lambda_1 \log(1-\theta),-\lambda_2 \log(1-\theta)) \vee \text{log series }(\theta) \underset{\theta}{\wedge} \text{Beta I }(a;\rho)$$

$$\sim \text{BGWD }(a;\lambda_1,\lambda_2;\rho)$$

$$\text{double Poisson }(\lambda_1\log(1+\theta),\lambda_2\log(1+\theta))\vee\text{log series}\left(\frac{\theta}{1+\theta}\right) \underset{\theta}{\wedge} \text{Beta II }(a;\rho)$$

$$\sim \text{BGWD }(a;\lambda_1,\lambda_2;\rho)$$

$$\text{Poisson }(\lambda\log(1-\theta_1-\theta_2)) \vee \text{biv.log series }(\theta_1,\theta_2) \underset{\theta_1,\theta_2}{\wedge} \text{biv. Beta I }(k,m;\rho)$$

$$\sim \text{BGWD }(\lambda;k,m;\rho)$$

$$\text{Poisson }(\lambda\log(1+\theta_1+\theta_2) \vee$$

$$\text{biv. log series}\left(\frac{\theta_1}{1+\theta_1+\theta_2},\ \frac{\theta_2}{1+\theta_1+\theta_2}\right) \underset{\theta_1,\theta_2}{\wedge} \text{Beta II }(k,m;\rho)$$

$$\sim \text{BGWD }(\lambda;k,m;\rho).$$

The logarithmic series distributions considered in the above models are defined by the following p.g.f.'s

$$g(s) = \log(1-\eta s)/\log(1-\eta),\ 0 < \eta < 1 \text{ (univariate case)}$$

and

$$g(s,t) = \log(1-\eta_1 s-\eta_2 t)/\log(1-\eta_1-\eta_2),\ 0 < \eta_i < 1,\ i = 1,2$$
$$\text{(bivariate case)}.$$

Sibuya (1980) provided an alternative unified version of the first and third models of this subsection in terms of mixing two independent Poisson distributions by what he calls a bivariate gamma product ratio distribution, i.e., he considered the following model:

double Poisson $(\lambda_1,\lambda_2) \bigwedge_{\lambda_1,\lambda_2}$ biv. gamma product ratio $(k,m;a;\rho)$

$\sim$ BGWD $(a;k,m;\rho)$

where the p.d.f. of the compounding distribution is

$$f(\lambda_1,\lambda_2) = \frac{\Gamma(a+\rho)}{\Gamma(k)\Gamma(m)\Gamma(a)\Gamma(\rho)} \lambda_1^{k-1} \lambda_2^{m-1} \int_0^{\infty} e^{-(\lambda_1+\lambda_2)t} t^{k+m+\rho-1} (1+t)^{-(a+\rho)} dt,$$

$$a,k,m,\rho > 0.$$

2.6. Mixtures of bivariate confluent hypergeometric distributions.

Let X,Y be nonnegative integer-valued r.v.'s such that their joint distribution has p.g.f. of the form

$$\begin{aligned} g(s,t) &= \frac{\Phi_2(k,m;d;\lambda s,\lambda t)}{\Phi_2(k,m;d;\lambda,\lambda)} \\ &= \frac{\Phi_2(k,m;d;\lambda s,\lambda t)}{{}_1F_1(k+m;d;\lambda)}, \quad k,m,d,\lambda > 0. \end{aligned} \tag{2.2}$$

Here Φ_2 is defined by

$$\Phi_2(a,b;c;x,y) = \sum_{r=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{a_{(r)} b_{(\ell)}}{c_{(r+\ell)}} \frac{x^r}{r!} \frac{y^\ell}{\ell!}$$

and ${}_1F_1$ can be obtained from

$$ {}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z) = \sum_{r=0}^{\infty} \frac{(a_1)_{(r)} \cdots (a_p)_{(r)}}{(b_1)_{(r)} \cdots (b_q)_{(r)}} \frac{z^r}{r!} \tag{2.3}$$

for $p = q = 1$. Assume further that λ is a r.v. having a distribution with p.d.f. given by

$$f(\lambda) = \frac{(a+1)^c \lambda^{c-1} e^{-(a+1)\lambda} {}_1F_1(k+m;d;\lambda)}{\Gamma(c)\ {}_2F_1(k+m,c;d;(a+1)^{-1})}, \quad \lambda,c > 0,\ a \geq 0$$

with ${}_2F_1$ obtained from (2.3) for $q = p/2 = 1$. (This is Kemp and Kemp's (1971) extension of Bhattacharya's (1966) generalized exponential distribution.) Then the resulting mixed distribution of (X,Y) has p.g.f. given by

$$G(s,t) = \frac{(a+1)^c \int_0^\infty \lambda^{c-1} e^{-(a+1)\lambda} \Phi_2(k,m;d;\lambda s;\lambda t)\, d\lambda}{\Gamma(c)\ {}_2F_1(k+m,c;d;(a+1)^{-1})}$$

$$= \frac{(a+1)^c}{\Gamma(c)\ {}_2F_1(k+m,c;d;(a+1)^{-1})} \sum_{r,\ell} \frac{{}^k(r)\,{}^m(\ell)}{{}^d(r+\ell)} \frac{s^r}{r!} \frac{t^\ell}{\ell!}$$

$$(2.4) \qquad \times \int_0^\infty e^{-(a+1)\lambda} \lambda^{c+r+\ell-1} d\lambda$$

$$= \frac{1}{{}_2F_1(k+m,c;d;(a+1)^{-1})} \sum_{r,\ell} \frac{{}^c(r+\ell)\,{}^k(r)\,{}^m(\ell)}{{}^d(r+\ell)} \frac{[s/(a+1)]^r}{r!} \frac{[t/(a+1)]^\ell}{\ell!}$$

$$= \frac{F_1(c;k,m;d;s(a+1)^{-1},t(a+1)^{-1})}{F_1(c;k,m;d;(a+1)^{-1},(a+1)^{-1})} .$$

The BGWD $(c;k,m;d-c-k-m)$ belongs to the family (2.2) for $a = 0$, $c,k,m,d > 0$ provided that $d-c-k-m > 0$. Hence, the BGWD $(c;k,m;d-c-k-m)$ can be considered as the mixture on λ of a bivariate distribution belonging to the family (2.2) if λ has a distribution with p.d.f.

$$f(\lambda) = e^{-\lambda} \lambda^{c-1} \frac{{}_1F_1(k+m;d;\lambda)}{{}_2F_1(c,k+m;d;1)\Gamma(c)} , \quad \lambda > 0,k,m,c,d > 0$$

provided that $d-c-k-m > 0$.

Thus,

$$\frac{1}{\Gamma(c)\,{}_2F_1(c,k+m;d;1)} \int_0^\infty \Phi_2(k,m;d;\lambda s,\lambda t)\lambda^{c-1} e^{-\lambda}\, d\lambda \sim \text{BGWD } (c;k,m;d-c-k-m).$$

Note that models 1.1, 2.2 and 2.4 have also been discussed by Janardan and Patil (1971), Xekalaki (1977) and Sibuya and Shimizu (1981). In addition, model 2.2 has been examined by Mosimann (1963).

3. *Conditionality Models.*

These are mixing models with a discrete mixing distribution.

3.1. *The bivariate negative hypergeometric conditionality model.*

Let X_1,X_2,Y_1,Y_2 be r.v.'s such that the conditional distribution of $(X_1,X_2)|(Y_1=y_1,Y_2=y_2)$ is the double negative hypergeometric with parameters (m,n,y_1), (h,ℓ,y_2) and p.d.f. given by

$$p_{x_1,x_2|y_1,y_2} = \binom{-m}{x_1}\binom{-n}{y_1-x_1}\binom{-h}{x_2}\binom{-\ell}{y_2-x_2}\Big/\binom{-m-n}{y_1}\binom{-h-\ell}{y_2}$$

where $m,n,h,\ell > 0$; $p_{x_1,x_2|y_1,y_2} = P(X_1=x_1, X_2=x_2 | Y_1=y_1, Y_2=y_2)$.

Let the distribution of (Y_1,Y_2) be the BGWD $(a;m+n,h+\ell;\rho)$. Then, the distribution of (X_1,X_2) is the BGWD $(a;m,h;\rho)$.

To prove the above statement, substitute for $p_{x_1,x_2|y_1,y_2}$ and p_{y_1,y_2} in the relationship

$$p_{x_1,x_2} = \sum_{y_1,y_2} p_{x_1,x_2|y_1,y_2}\, p_{y_1,y_2} .$$

Then

$$p_{x_1,x_2} = \frac{{}^{\rho}(m+n+h+\ell)}{(a+\rho)_{(m+n+h+\ell)}} \frac{{}^{m}(x_1)\,{}^{h}(x_2)}{x_1!\,x_2!}$$

$$\times \sum_{y_1=x_1}^{\infty} \sum_{y_2=x_2}^{\infty} \frac{{}^{n}(y_1-x_1)\,{}^{\ell}(y_2-x_2)\,{}^{a}(y_1+y_2)}{(\rho+a+m+n+h+\ell)_{(y_1+y_2)}(y_1-x_1)!\,(y_2-x_2)!}$$

$$= \frac{{}^{\rho}(m+n+\ell+h)\,{}^{m}(x_1)\,{}^{h}(x_2)\,{}^{a}(x_1+x_2)}{x_1!\,x_2!\,(a+\rho)_{(m+n+\ell+h)}(a+m+n+h+\ell+\rho)_{(x_1+x_2)}}$$

$$\times \sum_{y_1,y_2} \frac{(a+x_1+x_2)_{(y_1+y_2)}\,{}^{n}(y_1)\,{}^{\ell}(y_2)}{(a+m+n+h+\ell+\rho+x_1+x_2)_{(y_1+y_2)}} \frac{1}{y_1!} \frac{1}{y_2!}$$

$$= \frac{{}^{\rho}(m+n+\ell+h)\,{}^{m}(x_1)\,{}^{h}(x_2)\,{}^{a}(x_1+x_2)}{(a+\rho)_{(m+n+h+\ell+x_1+x_2)}} \frac{1}{x_1!} \frac{1}{x_2!} \frac{(\rho+m+h+a+x_1+x_2)_{(n+\ell)}}{(\rho+m+h)_{(n+\ell)}}$$

$$= \frac{{}^{\rho}(m+h)\,{}^{m}(x_1)\,{}^{h}(x_2)\,{}^{a}(x_1+x_2)}{(a+\rho)_{(m+h+x_1+x_2)}\,x_1!\,x_2!}$$

$$= \frac{{}^{\rho}(m+h)}{(a+\rho)_{(m+h)}} \frac{{}^{a}(x_1+x_2)\,{}^{m}(x_1)\,{}^{h}(x_2)}{(a+\rho+m+h)_{(x_1+x_2)}} \frac{1}{x_1!} \frac{1}{x_2!} .$$

Hence $(x_1,x_2) \sim$ BGWD $(a;m,h;\rho)$.

In the case $n = \ell = 1$ the converse of the result just proved is also true as indicated by the following model.

3.2. An identifiability property of the BGWD.

Let (X_1,X_2) and (Y_1,Y_2) be two random vectors such that the conditional distribution of $(X_1,X_2)|(Y_1=y_1,Y_2=y_2)$ is the double negative hypergeometric with p.f. given by

$$p_{x_1,x_2|y_1,y_2} = \binom{-m}{x_1}\binom{-1}{y_1-x_1}\binom{-h}{x_2}\binom{-1}{y_2-x_2}\Big/\binom{-m-1}{y_1}\binom{-h-1}{y_2}.$$

Then the distribution of (X_1,X_2) is the BGWD $(a;m+1,h+1;\rho)$ if and only if the distribution of (Y_1,Y_2) is the BGWD $(a;m,h;\rho)$.

The "if" part can be shown using an argument similar to that used in section 3.1. To prove the "only if" part, observe that

$$p_{r,\ell} = \sum_{n=r}^{\infty}\sum_{k=\ell}^{\infty} q_{n,k}\binom{m+n-1}{n}\binom{h+k-1}{k}\Big/\binom{m+n}{n}\binom{h+k}{k}, \quad \begin{array}{l} r = 0,1,2,\ldots \\ \ell = 0,1,2,\ldots \end{array}$$

where $p_{r,\ell} = P(X_1=r,X_2=\ell)$, $q_{n,k} = P(Y_1=n,Y_2=k)$. This is a functional equation in $q_{n,k}$. Since $(X_1,X_2) \sim$ BGWD $(a;m+1,h+1;\rho)$, it follows from model 3.1 that one solution for $q_{n,k}$ is the p.f. of the BGWD $(a;m,h;\rho)$. The uniqueness follows if one notes that

$$\sum_{n=r}^{\infty}\sum_{k=\ell}^{\infty} (q_{n,k}-q^*_{n,k})\Big/\binom{m+n}{n}\binom{h+k}{h} = 0; \quad \begin{array}{l} r = 0,1,2,\ldots \\ \ell = 0,1,2,\ldots \end{array}$$

where $q^*_{n,k}$ is another solution. If $H_{r,\ell}$ denotes the left hand side of this equation it follows that

$$H_{r,\ell}-H_{r+1,\ell}-H_{r,\ell+1}+H_{r+1,\ell+1} = 0; \quad \begin{array}{l} r = 0,1,2,\ldots \\ \ell = 0,1,2,\ldots, \end{array}$$

i.e., $q_{r,\ell} = q^*_{r,\ell}$, $r = 0,1,2,\ldots$, $\ell = 0,1,2,\ldots$.

3.3. The shifted bivariate generalized Waring conditionality model.

Let X_1, X_2, Y_1, Y_2 be discrete r.v.'s such that the conditional distribution of $(X_1,X_2)|(Y_1=y_1,Y_2=y_2)$ is the BGWD $(a+y_1+y_2;n,\ell;\rho+m+h)$ shifted y_1 units in the direction of x_1 and y_2 units in the direction of x_2 where $a,n,\ell,m,h,\rho > 0$, i.e., the vector $(X_1,X_2)|(Y_1=y_1,Y_2=y_2)$ has p.g.f.

$$G_{X_1,X_2|y_1,y_2}(s,t) = \frac{\rho_{(n+\ell)}}{(a+\rho+y_1+y_2)_{(n+\ell)}} \times F_1(a+y_1+y_2;n,\ell;a+n+\ell+\rho+m+h+y_1+y_2;s,t). \tag{3.1}$$

Let the distribution of (Y_1,Y_2) be the BGWD $(a;m,h;\rho)$. Then the distribution of (X_1,X_2) is the BGWD $(a;m+n,h+\ell;\rho)$.

Proof. Let $G_{X_1,X_2}(s,t)$ be the p.g.f. of (X_1,X_2). Then using (3.1) and substituting for p_{y_1,y_2}, $G_{X_1,X_2|y_1,y_2}(s,t)$ in the formula

$$G_{X_1,X_2}(s,t) = \sum_{y_1,y_2} G_{X_1,X_2|y_1,y_2}(s,t)\, p_{y_1,y_2}$$

we get

$$G_{X_1,X_2}(s,t) = \rho_{(m+h+n+\ell)} \times \sum_{y_1,y_2} \frac{s^{y_1}t^{y_2}F_1(a+y_1+y_2;n,\ell;a+y_1+y_2+n+\ell+m+h+\rho;s,t)\,a_{(y_1+y_2)}m_{(y_1)}h_{(y_2)}}{(a+\rho)_{(m+n+h+\ell+y_1+y_2)}\,y_1!\,y_2!}$$

$$= \frac{\rho_{(m+n+h+\ell)}}{(a+\rho)_{(m+n+h+\ell)}} \times \sum_{x_1,x_2}\sum_{y_1,y_2} \frac{a_{(x_1+x_2+y_1+y_2)}m_{(y_1)}h_{(y_2)}n_{(x_1)}\ell_{(x_2)}\,s^{x_1+y_1}t^{x_2+y_2}}{(a+\rho+m+n+h+\ell)_{(x_1+x_2+y_1+y_2)}\,x_1!\,x_2!\,y_1!\,y_2!}$$

$$= \frac{\rho_{(m+n+h+\ell)}}{(a+\rho)_{(m+n+h+\ell)}}\; F_D(a;n,m,\ell,h;a+\rho+m+n+h+\ell;s,s,t,t)$$

where F_D is obtained for $r = 4$ from Lauricella's r-variate generalized hypergeometric series of the fourth type defined by

$$F_D(a;b_1,b_2,\ldots,b_r;c;x_1,x_2,\ldots,x_r)$$

$$= \sum_{\substack{0\le m_i<\infty\\ 1\le i\le r}} \frac{a^{(m_1+m_2+\ldots+m_r)}(b_1)^{(m_1)}(b_2)^{(m_2)}\cdots(b_r)^{(m_r)}}{c^{(m_1+m_2+\ldots+m_r)}}$$

$$\times \frac{x_1^{m_1}}{(m_1)!}\frac{x_2^{m_2}}{(m_2)!}\cdots\frac{x_r^{m_r}}{(m_r)!}, \quad |x_i| < 1,\ i = 1,2,\ldots,r,$$

i.e.,

$$G_{X_1,X_2}(s,t) = \frac{\rho^{(m+n+h+\ell)}}{(a+\rho)^{(m+n+h+\ell)}} F_1(a;m+n,h+\ell;a+\rho+m+n+h+\ell;s,t).$$

Therefore, $(X_1,X_2) \sim$ BGWD $(a;m+n,h+\ell;\rho)$.

3.4. The bivariate inverse hypergeometric conditionality model.

Let $(X_1,X_2),(Y_1,Y_2)$ be two random vectors with nonnegative integer-valued components. Assume that the conditional distribution of (X_1,X_2) given $(Y_1=y_1,Y_2=y_2)$ is the bivariate inverse hypergeometric with p.f.

$$(3.2)\qquad p_{x_1,x_2|y_1,y_2} = \frac{B(b+x_1+x_2,c+y_1+y_2-x_1-x_2)}{B(b,c)}\binom{y_1}{x_1}\binom{y_2}{x_2},$$

$$b,c > 0;\ x_i = 0,1,\ldots,y_i;\ i = 1,2$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$. Assume further that the distribution of (Y_1,Y_2) is the BGWD $(a;k,m;\rho)$. Then the distribution of (X_1,X_2) is the BGWD $(b;k,m;\rho)$ (Janardan, 1973).

3.5. The BGWD as a posterior distribution.

Consider two random vectors (X_1,X_2), (Y_1,Y_2) with nonnegative integer-valued components and let $(X_1,X_2)|(Y_1=y_1,Y_2=y_2)$ be distributed as in (3.2). Suppose further that the distribution of (Y_1,Y_2) is the BGWD $(a;k;m;\rho)$. Then the distribution of the random vector $(Y_1-X_1,Y_2-X_2)|(X_1=x_1,X_2=x_2)$ is the BGWD $(c;k+x_1,m+x_2;b+\rho)$ (Janardan, 1973).

4. *Miscellaneous Derivations.*

4.1. *The STER model.*

Let $(X_1,X_2),(Y_1,Y_2)$ be random vectors with nonnegative integer-valued components. Assume that their p.f.'s $p_{r,\ell} \equiv P(X_1=r,X_2=\ell)$ and $q_{r,\ell} \equiv P(Y_1=r,Y_2=\ell)$ satisfy the relationship

(4.1)
$$q_{r,\ell} = c\sum_{x=r+1}^{\infty}\sum_{y=\ell+1}^{\infty}\frac{p_{x,y}}{x+y}, \quad \begin{array}{l} r = 0,1,2,\ldots \\ \ell = 0,1,2,\ldots \end{array}$$

where c is the normalizing constant. (The model (4.1) is a bivariate version of Bissinger's (1965) STER model.) Then the distribution of (Y_1,Y_2) is the BGWD $(2;1,1;\rho)$ if and only if the distribution of (X_1,X_2) is the BGWD $(1;1,1;\rho)$ (Xekalaki, 1983c).

4.2. *The BGWD as the only distribution with tail probabilities satisfying a certain condition.*

Assume that (X_1,X_2) is a vector of nonnegative integer-valued components. Then

$$P(X_1>r,X_2=\ell) = P(X_1=r,X_2>\ell) = (ar+b\ell+1)P(X_1=r,X_2=\ell),$$
$$r,\ell = 0,1,2,\ldots;\ 0<a<1,\ b>0$$

if and only if the distribution of (X_1,X_2) is the BGWD $(\frac{b}{a};1,;\frac{1}{a}-1)$ (Xekalaki, 1983c).

4.3. *The exceedance model.*

Suppose that $(Y_1,Y_2,\ldots,Y_n)$ is a random sample from some population with a continuous distribution and let $Y_{(1)} \le Y_{(2)} \le \ldots \le Y_{(n)}$ be the corresponding order statistics. Let j be a fixed integer number, $1 \le j \le n$. Consider sampling once more from the same population till the number of the new observations becomes equal to k. Let $(Z_1, Z_2, \ldots, Z_k)$ be the new sample $(n \geq k)$. Let i be another integer number, $1 \leq i \leq j \leq n$. Denote by X the number of Z's that are less than $Y_{(i)}$ and by Y the number of Z's that are between $Y_{(i)}$ and $Y_{(j)}$. Then the distribution function of (X,Y) is the bivariate negative binomial with p.f.

$$\frac{\Gamma(k+x+y)}{\Gamma(k)x!y!} p^x q^y (1-p-q)^k, \quad x,y = 0,1,2,\ldots$$

where the vector

$$(p,q) \equiv (P(Z < Y_{(i)}), \ P(Z < Y_{(j)}) - P(Z < Y_{(i)}))$$

follows the bivariate beta I distribution with p.d.f.

$$\frac{\Gamma(h+1)}{\Gamma(i)\Gamma(j-i)\Gamma(h-j+1)} p^{i-1} q^{j-i-1} (1-p-q)^{h-j}, \quad h-j+1 > 0 .$$

Then the resulting distribution of (X,Y) is the BGWD $(k;i,j;h-j-i+1)$ (Sibuya and Shimizu, 1981).

5. *The BGWD as a Member of the Bivariate Generalization of the Pearson System of Discrete Distributions.*

Many attempts were made to extend the distributions of the univariate Pearson system to bivariate distributions. Van Uven (1947,1948) investigated distributions whose p.d.f.'s satisfied

$$\frac{1}{f(x,y)} \frac{\partial f(x,y)}{\partial x} = \frac{L_1(x,y)}{Q_1(x,y)}$$
$$\frac{1}{f(x,y)} \frac{\partial f(x,y)}{\partial y} = \frac{L_2(x,y)}{Q_2(x,y)} \tag{5.1}$$

where L_1, L_2 are linear functions in x,y, Q_1, Q_2 are quadratic functions in x,y, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

The system (5.1) can be obtained as the limit of the following pair of difference equations as the lattice width goes to zero.

$$(5.2)\quad \begin{aligned} \frac{\Delta_x f(x-1,y)}{f(x-1,y)} &= \frac{a_0+a_1x+a_2y}{b_0+b_1x+b_2y+b_3x(x-1)+b_4y(y-1)+b_{34}xy} \\ \frac{\Delta_y f(x,y-1)}{f(x,y-1)} &= \frac{a_0'+a_1'x+a_2'y}{b_0'+b_1'x+b_2'y+b_3'x(x-1)+b_4'y(y-1)+b_{34}'xy} \end{aligned}$$

where

$$\Delta_x f(x-1,y) = f(x,y) - f(x-1,y)$$

$$\Delta_y f(x,y-1) = f(x,y) - f(x,y-1).$$

This system of equations defines what is known in the literature as the Pearson system of bivariate discrete distributions (see Ord, 1972). The BGWD can be shown to belong to the family of bivariate discrete distributions defined by the equations (5.2). Indeed, its probabilities satisfy the following recurrence relations.

$$\frac{p_{i,j}}{p_{i-1,j}} = \frac{(a+i+j-1)(k+i-1)}{(a+k+\rho+m+i+j-1)i}, \qquad \begin{array}{l} i = 1,2,\ldots \\ j = 0,1,2,\ldots \end{array}$$

and

$$\frac{p_{i,j}}{p_{i,j-1}} = \frac{(a+i+j-1)(m+j-1)}{(a+k+\rho+m+i+j-1)j}, \qquad \begin{array}{l} i = 0,1,2,\ldots \\ j = 1,2,\ldots\ . \end{array}$$

Therefore

$$(5.3)\quad \begin{aligned} \frac{\Delta_i p_{i-1,j}}{p_{i-1,j}} &= \frac{(a-1)(k-1)-(\rho+m+1)i+(k-1)j}{(a+k+\rho+m)i+i(i-1)+ij}, \quad \begin{array}{l} i = 1,2,\ldots \\ j = 0,1,2,\ldots \end{array} \\ \frac{\Delta_j p_{i,j-1}}{p_{i,j-1}} &= \frac{(a-1)(m-1)-(\rho+k+1)j+(m-1)i}{(a+k+\rho+m)j+j(j-1)+ij}, \quad \begin{array}{l} i = 0,1,2,\ldots \\ j = 1,2,\ldots\ . \end{array} \end{aligned}$$

Hence, the BGWD $(a;k,m;\rho)$ is a solution of (5.2) for $a_0 = (a-1)(k-1)$, $a_0' = (a-1)(m-1)$, $a_1 = -(\rho+m+1)$, $a_1' = -(\rho+k+1)$, $a_2 = k-1$, $a_2' = m-1$, $b_0 = b_0' = b_2 = b_4 = b_1' = b_3' = 0$, $b_1 = b_2' = a+k+\rho+m$, $b_3 = b_4' = b_{34} = b_{34}' = 1$.

6. Relation of the BGWD to the Pearson's System of Continuous Distributions.

Irwin (1975) determined a frequency curve to approximate the UGWD by equating its slope-to-mean ordinate ratio to $\frac{1}{f}\frac{df}{dx}$, where f was the p.d.f. of a distribution belonging to Pearson's system. He called the curve "the continuous analogue of the UGWD and pointed out that in general it is of Type VI (Beta II).

We now provide a bivariate extension of the slope-to-mean ordinate method with the aim of determining the continuous analogue of the BGWD.

Let the probabilities of a distribution on $\{0,1,2,\ldots\} \times \{0,1,2,\ldots\}$ be $p_{r,\ell}$, $r = 0,1,2,\ldots$, $\ell = 0,1,2,\ldots$. Define the ratio of the slope in the direction of r to the mean ordinate at the point $(r-\frac{1}{2},\ell)$ by

$$R(r-\tfrac{1}{2},\ell) = \frac{p_{r,\ell}-p_{r-1,\ell}}{\frac{1}{2}\{p_{r,\ell}+p_{r-1,\ell}\}} \quad (6.1)$$

Similarly, define the ratio of the slope in the direction of ℓ to the mean ordinate at the point $(r,\ell-\frac{1}{2})$ by

$$L(r,\ell-\tfrac{1}{2}) = \frac{p_{r,\ell}-p_{r,\ell-1}}{\frac{1}{2}\{p_{r,\ell}+p_{r,\ell-1}\}} \quad (6.2)$$

Then, the continuous analogue $\phi(x,y)$ of the probability distribution $p_{x,y}$ can be defined to be the distribution satisfying the equations

$$\frac{1}{\phi(r-\frac{1}{2},\ell)}\frac{\partial}{\partial x}\phi(r-\tfrac{1}{2},\ell) = R(r-\tfrac{1}{2},\ell)$$

$$\frac{1}{\phi(r,\ell-\frac{1}{2})}\frac{\partial}{\partial y}\phi(r,\ell-\tfrac{1}{2}) = L(r,\ell-\tfrac{1}{2})$$

or, equivalently,

$$\frac{1}{\phi(r,\ell)} \frac{\partial}{\partial X} \phi(r,\ell) = R(r,\ell)$$

(6.3)

$$\frac{1}{\phi(r,\ell)} \frac{\partial}{\partial Y} \phi(r,\ell) = L(r,\ell)$$

where $X = x + \frac{1}{2}$ and $Y = y + \frac{1}{2}$.

For the BGWD $(a;k,m;\rho)$, using (6.1), (6.2) and (5.3) we obtain

$$R(r-\tfrac{1}{2},\ell) = \frac{(a-1)(k-1) - (\rho+m+1)r + (k-1)\ell}{r^2+r\ell+[a+k+ \frac{\rho+m-3}{2}]r + \frac{k-1}{2}\ell+\frac{(a-1)(k-1)}{2}}$$

(6.4)

$$L(r,\ell-\tfrac{1}{2}) = \frac{(a-1)(m-1) - (\rho+k+1)\ell + (m-1)r}{\ell^2+r\ell + [a+m+ \frac{\rho+k-3}{2}]\ell+ \frac{m-1}{2} r+ \frac{(a-1)(m-1)}{2}} .$$

Therefore, from (6.3), (6.4) it follows that the continuous analogue is the distribution satisfying the equations

$$\frac{1}{\phi}\frac{\partial\phi}{\partial X} = \frac{(a-1)(k-1) - (\rho+m+1)X+(k-1)Y}{X^2+XY+[a+k+ \frac{\rho+m-3}{2}] X+ \frac{k-1}{2} Y+\frac{(a-1)(k-1)}{2}}$$

(6.5)

$$\frac{1}{\phi}\frac{\partial\phi}{\partial Y} = \frac{(a-1)(m-1) - (\rho+k+1)Y + (m-1)X}{Y^2+XY+[a+m+ \frac{\rho+k-3}{2}]Y + \frac{m-1}{2} X+ \frac{(a-1)(m-1)}{2}} .$$

Obviously ϕ belongs to the Pearson system of continuous bivariate distributions defined by (5.1). Its exact form cannot be determined since the roots of the denominator in equations (6.5) are not rational functions of X and Y. However, it can be seen that the conditional distribution of X given Y and that of Y given X are members of the univariate Pearson system and are of type VI (beta II). Specifically,

$$X|(Y=y) \sim C_1(X+a_1(y))^{-q_1(y)}(X+a_2(y))^{-q_2(y)}$$

and

(6.6) $$Y|(X=x) \sim C_2(Y+b_1(x))^{-p_1(x)}(Y+b_2(x))^{-p_2(x)}$$

where

$$C_1 = \frac{\Gamma(q_1(y))[a_1(y)-a_2(y)]^{q_2(y)-q_1(y)+1}}{\Gamma(q_2(y)+1)\Gamma(q_1(y)-q_2(y)-1)}$$

$$C_2 = \frac{\Gamma(p_1(x))[b_1(x)-b_2(x)]^{p_2(x)-p_1(x)+1}}{\Gamma(p_2(x)+1)\Gamma(p_1(x)-p_2(y)-1)}$$

$$q_i(y) = \frac{(k-1)(a+y-1)+(\rho+m+1)a_i(y)}{a_1(y)-a_2(y)}, \qquad a_1(y) > a_2(y), \quad i = 1,2$$

$$p_i(x) = \frac{(m-1)(a+x-1)+(\rho+k+1)b_i(x)}{b_1(x)-b_2(x)}, \qquad b_1(x) > b_2(x), \quad i = 1,2$$

and $-a_i(y)$, $i = 1,2$ are the roots of the denominator of the first of equations (6.5) which has been considered as a polynomial in X. The coefficients $b_i(x)$, $i = 1,2$ are defined accordingly.

It would seem possible to obtain the continuous analogue of the BGWD explicitly if the slope-to-mean ordinate ratios as given by (5.3) were used instead of (6.1) and (6.2). In that case we would have the continuous analogue to be the solution of the following system of differential equations

(6.7)
$$\frac{1}{\phi(x,y)}\frac{\partial}{\partial x}\phi(x,y) = \frac{(a-1)(k-1)-(\rho+m+1)x+(k-1)y}{x(x+y+a+k+\rho+m-1)}$$

$$\frac{1}{\phi(x,y)}\frac{\partial}{\partial y}\phi(x,y) = \frac{(a-1)(m-1)-(\rho+k+1)y+(m-1)x}{y(x+y+a+k+\rho+m-1)} .$$

But, because $\rho+m+k > 0$, it is not possible to integrate equations (6.7) unless $k = m = 1$ whence the continuous analogue is of the form

$$\phi(x,y) = C(x+y+a+\rho+1)^{-(\rho+2)}$$

where $C = (a+\rho+1)^{\rho}\frac{\Gamma(\rho+2)}{\Gamma(\rho)}$. This is bivariate Pearson type IIaβ (bivariate Beta II $(1,1;\rho)$ or Dirichlet type II).

In the more general case $k,m \neq 1$ we can only make inference about the form of the conditional distributions of $X|(Y=y)$ and $Y|(X=x)$. In particular,

$$X|(Y=y) \sim Cx^p(x+a+k+m+\rho+y-1)^{-(p+\rho+m+1)}$$

where

$$C = (a+k+m+\rho+y-1)^{\rho+m} \frac{\Gamma(p+\rho+m+1)}{\Gamma(p+1)\Gamma(\rho+m)}$$

$$p = \frac{(k-1)(a+y-1)}{a+k+m+\rho+y-1} .$$

A similar expression holds for the p.d.f. of $Y|(X=x)$.

7. Limiting Cases of the BGWD.

The purpose in this section is to provide limiting forms for the BGWD.

It is well known that

$$\lim_{n\to+\infty} \frac{\Gamma(n)n^{\alpha}}{\Gamma(\alpha+n)} = 1$$

where α is a positive real number and n a positive integer. This result can be extended over positive real values of n as follows.

Let β be a positive real number. Then,

(7.1) $$\lim_{\beta\to\infty} \frac{\Gamma(\beta)\beta^{\alpha}}{\Gamma(\alpha+\beta)} = \lim_{\beta\to\infty} \frac{(\frac{\beta}{\beta+\alpha})^{\alpha-\frac{1}{2}}e^{\alpha}}{(1+\frac{\alpha}{\beta})^{\beta}} = 1.$$

Here we made use of the known result (e.g., see Erdélyi et al., (1953), Vol. 1, p. 47) that for large β ,

$$\Gamma(\beta) = (2\pi)^{\frac{1}{2}} e^{-\beta} \beta^{\beta-\frac{1}{2}} \left(1+\frac{\beta^{-1}}{12}+\frac{\beta^{-2}}{288}-\frac{139\beta^{-3}}{51840}-0(z^{-4})\right) .$$

THEOREM 7.1. *The BGWD* $(a;k,m;\rho)$ *tends to the double negative binomial distribution* $(k,m;\frac{a}{a+\rho},\frac{a}{a+\rho})$ *if* $\rho\to+\infty$, $a\to+\infty$ *while* $\frac{a}{a+\rho}<1$ *and* k,m *are positive constants.*

Proof. Let H denote the set of conditions $a\to+\infty$, $\rho\to+\infty$, $\frac{a}{a+\rho}<1$, k,m positive constants. Then, using (7.1),

$$\lim_{H} \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} \frac{a_{(x+y)}k_{(x)}m_{(y)}}{(a+k+m+\rho)_{(x+y)}x!y!}$$

$$= \lim_{H}\left(\frac{\rho}{a+\rho}\right)^{k+m} \lim_{H}\left(\frac{a}{a+\rho}\right)^{x+y} \frac{(1+\frac{1}{a})\ldots(1+\frac{x+y-1}{a})k_{(x)}m_{(y)}}{(1+\frac{k+m}{a+\rho})\ldots(1+\frac{k+m+x+y-1}{a+\rho})x!y!}$$

$$= \frac{k_{(x)}}{x!}\frac{m_{(y)}}{y!}\left(\frac{\rho}{a+\rho}\right)^{k+m}\left(\frac{a}{a+\rho}\right)^{x+y}$$

which implies that

$$\lim_{H} \text{BGWD}\ (a;k,m;\rho) = \frac{\Gamma(k+x)}{\Gamma(k)x!}\frac{\Gamma(m+y)}{\Gamma(m)y!}\left(\frac{\rho}{a+\rho}\right)^{k+m}\left(\frac{a}{a+\rho}\right)^{x+y} .$$

Hence the theorem is established.

THEOREM 7.2. *The BGWD* $(a;k,m;\rho)$ *tends to the double Poisson distribution with parameters* $\frac{ak}{a+\rho}$, $\frac{am}{a+\rho}$ *if* $a \to +\infty$, $k \to +\infty$, $m \to +\infty$, $\rho \to +\infty$ *while* $\frac{a}{a+\rho} \to 0$, $\frac{ak}{a+\rho} < +\infty$, $\frac{am}{a+\rho} < +\infty$.

Proof. Let H' denote the set of conditions $a \to +\infty$, $k \to +\infty$, $m \to +\infty$, $\rho \to +\infty$, $\frac{a}{a+\rho} \to 0$, $\frac{ak}{a+\rho} < +\infty$, $\frac{am}{a+\rho} < +\infty$. Then

(7.2)
$$\lim_{H'} \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} \frac{a_{(x+y)}k_{(x)}m_{(y)}}{(a+k+m+\rho)_{(x+y)}} \frac{1}{x!}\frac{1}{y!}$$
$$= \lim_{H'} \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} \lim_{H'} \frac{a_{(x+y)}k_{(x)}m_{(y)}}{(a+k+m+\rho)_{(x+y)}x!y!} .$$

From (7.1) we have

$$\lim_{H'} \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} = \lim_{H'} \left(\frac{\rho}{a+\rho}\right)^{k+m}$$

$$(7.3) \qquad = \lim_{H'} e^{(k+m)\log \frac{\rho}{a+\rho}}$$

$$= \lim_{H'} e^{-\frac{(k+m)a}{a+\rho} \frac{a+\rho}{a} \log \frac{a+\rho}{\rho}}$$

$$= e^{-\frac{(k+m)a}{a+\rho} \lim\limits_{H'} \frac{a+\rho}{a} \log \frac{a+\rho}{\rho}} .$$

Note that

$$1 \le \frac{a+\rho}{a} \log \frac{a+\rho}{\rho} \le \frac{a}{\rho} \frac{a+\rho}{a} .$$

Taking limits under assumptions H' we get

$$1 \le \lim_{H'} \frac{a+\rho}{a} \log \frac{a+\rho}{\rho} \le 1$$

which implies that

$$(7.4) \qquad \lim_{H'} \frac{a+\rho}{a} \log \frac{a+\rho}{\rho} = 1.$$

Using (7.3) and (7.4) we have

$$(7.5) \qquad \lim_{H'} \frac{\rho}{a+\rho}^{k+m} = e^{-\frac{(k+m)a}{a+\rho}}$$

Moreover,

(7.6)

$$\lim_{H'} \frac{a_{(x+y)} k_{(x)} m_{(y)}}{(a+k+m+\rho)_{(x+y)}} \frac{1}{x!} \frac{1}{y!}$$

$$= \lim_{H'} \left(\frac{a}{a+\rho}\right)^{x+y} \frac{k^x}{x!} \frac{m^y}{y!} \frac{(1+\frac{1}{a})\dots(1+\frac{x+y-1}{a})(1+\frac{1}{k})\dots(1+\frac{x-1}{k})(1+\frac{1}{m})\dots(1+\frac{y-1}{m})}{(1+\frac{1}{a+\rho})\dots(1+\frac{k+m+x+y-1}{a+\rho})}$$

$$= \left(\frac{a}{a+\rho}\right)^{x+y} \frac{k^x}{x!} \frac{m^y}{y!}$$

Then, from (7.2), (7.5), (7.6) we obtain

$$\text{(7.7)} \qquad \lim_{H'} \text{BGWD}\,(a;k,m;\rho) = e^{-\frac{(k+m)a}{a+\rho}} \left(\frac{ak}{a+\rho}\right)^{x} \left(\frac{am}{a+\rho}\right)^{y} \Big/ x!y!$$

which establishes the result.

With the assumption that the scale is at our choice the following theorems are proved.

THEOREM 7.3. *Letting* $x \to \frac{a}{c_1}x$, $y \to \frac{a}{c_2}y$, c_1, c_2 *positive constants the BGWD* $(a;k,m;\rho)$ *tends to the bivariate Beta II distribution* $(k,m;\rho)$ *if* $a \to +\infty$.

Proof. We have

$$\lim_{a\to+\infty} \frac{a^2}{c_1c_2} \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} \frac{a_{(\frac{a}{c_1}x+\frac{a}{c_2}y)}\, k_{(\frac{a}{c_1}x)}\, m_{(\frac{a}{c_2}y)}}{(a+k+m+\rho)_{(\frac{a}{c_1}x+\frac{a}{c_2}y)}} \frac{1}{(\frac{a}{c_1}x)!} \frac{1}{(\frac{a}{c_2}y)!}$$

$$= \lim_{a\to+\infty} \frac{a^2}{c_1c_2} \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} \frac{a_{(k+m+\rho)}\, k_{(\frac{ax}{c_1})}\, m_{(\frac{ay}{c_2})}}{[a+a(\frac{x}{c_1}+\frac{y}{c_2})]_{(k+m+\rho)}\, 1_{(\frac{ax}{c_1})}\, 1_{(\frac{ay}{c_2})}}$$

$$= \lim_{a\to+\infty} \frac{a^2}{c_1c_2} \frac{\rho_{(k+m)}}{(a+\rho)_{(k+m)}} \frac{a_{(k+m+\rho)}}{[a(1+\frac{x}{c_1}+\frac{y}{c_2})]_{(k+m+\rho)}} \frac{(1+\frac{ax}{c_1})_{(k-1)}}{\Gamma(k)} \frac{(1+\frac{ay}{c_2})_{(m-1)}}{\Gamma(m)}$$

$$= (c_1c_2)^{-1} \frac{\rho(k+m)}{\Gamma(k)\Gamma(m)} \left(\frac{x}{c_1}\right)^{k-1} \left(\frac{y}{c_2}\right)^{m-1} \left(1+\frac{x}{c_1}+\frac{y}{c_2}\right)^{-(k+m+\rho)}$$

which implies that

$$\lim_{\substack{x\to ax/c_1 \\ y\to ay/c_2 \\ a\to+\infty}} \text{BGWD}\,(a;k,m;\rho) = (c_1c_2)^{-1} \frac{\Gamma(k+m+\rho)}{\Gamma(k)\Gamma(m)\Gamma(\rho)} \left(\frac{x}{c_1}\right)^{k-1} \left(\frac{y}{c_2}\right)^{m-1} \left(1+\frac{x}{c_1}+\frac{y}{c_2}\right)^{-(k+m+\rho)}.$$

THEOREM 7.4. *Letting* $x \to \frac{ax}{c_1\rho}$, $y \to \frac{ay}{c_2\rho}$ *with* c_1, c_2 *as finite positive constants, the BGWD* $(a;k,m;\rho)$ *tends to the uncorrelated bivariate Gamma distribution with parameters* $(k,m;1,1)$ *if we let* $a \to +\infty$ *first and then* $\rho \to +\infty$.

Proof. Applying theorem 7.3 we have

$$\lim_{\rho\to+\infty} \Big(\lim_{\substack{x\to ax/c_1\rho \\ y\to ay/c_2\rho \\ a\to+\infty}} \text{BGWD}\,(a;k,m;\rho)$$

$$= \lim_{\rho\to+\infty} \frac{1}{\rho^2 c_1 c_2} \frac{\Gamma(k+m+\rho)}{\Gamma(k)\Gamma(m)\Gamma(\rho)} \left(\frac{x}{c_1\rho}\right)^{k-1} \left(\frac{y}{c_2\rho}\right)^{m-1} \left(1+\frac{x}{c_1\rho}+\frac{y}{c_2\rho}\right)^{-(k+m+\rho)}$$

$$= \left(\frac{x}{c_1}\right)^{k-1} \left(\frac{y}{c_2}\right)^{m-1} \frac{(c_1c_2)^{-1}}{\Gamma(k)\Gamma(m)} \lim_{\rho\to+\infty} \rho_{(k+m)}\, \rho^{-k-m} \left(1+\frac{x}{c_1\rho}+\frac{y}{c_2\rho}\right)^{-(k+m+\rho)}$$

But

$$\lim_{\rho\to+\infty} \rho_{(k+m)}\, \rho^{-k-m} \left(1+\frac{x}{c_1\rho}+\frac{y}{c_2\rho}\right)^{-(k+m+\rho)} = \lim_{\rho\to+\infty} \left(1+\frac{x}{c_1\rho}+\frac{y}{c_2\rho}\right)^{-\rho} = e^{-\frac{x}{c_1}-\frac{y}{c_2}}.$$

Therefore, from (7.8) we have that

$$\lim_{\substack{x\to ax/c_1\rho \\ y\to ay/c_2\rho \\ a\to+\infty}} \big(\lim_{\rho\to+\infty} \text{BGWD}\,(a;k,m;\rho)\big) = \frac{(c_1c_2)^{-1}}{\Gamma(k)\Gamma(m)} \left(\frac{x}{c_1}\right)^{k-1} \left(\frac{y}{c_2}\right)^{m-1} e^{-\frac{x}{c_1}-\frac{y}{c_2}} .$$

Thus the theorem is established.

It may be noticed that results analogous to those obtained in this section have been noted, for the univariate case, by Irwin (1975).

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