Stochastic Processes and their Applications 29 (1988) 267-279
North-Holland


# THE CONTINUITY OF THE QUADRATIC VARIATION OF TWO-PARAMETER MARTINGALES

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Received 8 September 1987
Revised 31 March 1988

It has been known that any $L\log^+L$-integrable two-parameter martingale $M$ possesses a quadratic variation $[M]$. We show that the continuity properties of $M$ are inherited by its quadratic variation. If $M$ has no point jumps, $[M]$ has no point jumps. $[M]$ has at most axial jumps with respect to one of the coordinate axes in parameter space if $M$ possesses this property. Finally, $[M]$ is continuous along with $M$.

*AMS 1980 Subject Classifications*: Primary 60G48, 60G07; Secondary 60G60, 60G44.

*Keywords*: two-parameter martingales * quadratic variation * point jumps * axial jumps * continuity

## Introduction

It has been known for some time that square integrable continuous two-parameter martingales possess a quadratic variation (see Nualart [11]). On the basis of this knowledge, the fundaments of a stochastic calculus for martingales of this kind have been established (see Nualart [12]). The investigation of quadratic variation as one of the key notions of stochastic analysis has recently, in [9], been extended in two different directions. Firstly, the continuity requirement could be dropped. Square integrable martingales are seen to possess a unique orthogonal decomposition by three jump components corresponding to the three possible versions of jumps of the first kind that functions of two variables can have, point jumps and axial jumps propagating along parallels of the two axes, and by a continuous component. The quadratic variation of square integrable martingales can then be obtained as just the sum of the quadratic variations of their four components. Secondly, the integrability assumption was reduced to the "natural" degree. An inequality linking the square sums, which approximate the quadratic variation, uniformly to the $L\log^+L$-norm of a martingale allows one to extend the existence theorem for quadratic variation to martingales which are $L\log^+ L$-integrable. Moreover, a counterexample exhibiting a martingale which is not $L\log^+ L$-integrable and possesses no quadratic variation, suggests that $L\log^+ L$-integrability characterizes the space of two-parameter martingales for which it exists.

Of course, the quadratic variation $[M]$ of a martingale $M$ is expected to have the same continuity properties as $M$ itself. In [11], Nualart proved that if $M$ is square integrable, $[M]$ inherits continuity. In [9], this result could be extended. It has been shown that together with $M$ its quadratic variation possesses no $i$-jumps and that $[M]$ has at most $i$-jumps if $M$ has, $i=0,1,2$. Here the "0" stands for point jumps, whereas the "1" or "2" indicates axial jumps along the respective axis. In the proofs of these statements, the above mentioned orthogonal decomposition of square integrable martingales played an essential role. Therefore the results could only be established for this class of martingales. Nothing seemed to be known so far about the continuity properties of the quadratic variation of $L\log^+ L$-integrable martingales outside $L^2$.

In the present paper, we will fill this gap. Let $M$ be an $L\log^+ L$-integrable martingale. According to the theorem of Bakry, Millet and Sucheston [2, 10] we may assume that $M$ is regular, i.e. its trajectories are continuous for approach from the right upper quadrant and possess limits for approach from the remaining three. In Theorem 1 we will show that $[M]$ has no 0-jumps if $M$ does, in Theorem 2 that together with $M$ its quadratic variation has at most $i$-jumps, $i=1,2$. The most difficult problem turned out to be the proof of the fact that $[M]$ is continuous along with $M$. This will be presented in Theorem 3.

The methods employed to obtain these results are suggested by what has already been known. Roughly speaking, we approximate $M$, the "large" martingale, by a sequence $(M^n)_{n\in\mathbb{N}}$ of "small" martingales. Here "small" stands for "bounded" or at least "almost bounded", i.e. $p$-integrable for all $p$. We apply our knowledge of the continuity properties of the quadratic variations of the approximations and a simple proposition stating that $[M^n]$ converges to $[M]$ uniformly in probability. One difficulty which has to be faced with this approach lies in the fact that the approximations $M^n$ may not have the same continuity properties as $M$ itself. This problem is overcome mainly by showing that the jump parts of $M^n$ corresponding to the jump kinds which $M$ does not possess, are in some sense "orthogonal" to $M$ and thus "die out" as $n$ goes to infinity.

## 0. Notation, definitions and basics

The stochastic processes considered in this paper are parametrized by $\mathrm{I}=[0,1]$ or $\mathbb{I}=[0,1]^2$. The unit square is ordered by "$\leq$", which is understood to be coordinatewise linear ordering on I. Intervals with respect to this partial ordering are defined as usual. If $J$ is an interval, we write $s^J$, $t^J$ for the respective upper and lower corners. By a partition of a parameter interval we always mean a partition generated by a finite number of axial parallel lines (points) consisting of left open, right closed intervals (in the relative topology of $\mathbb{I}$ (I)). A 0-sequence of partitions is a sequence of partitions which increases with respect to fineness and the mesh of which goes to 0. To denote components of points in $\mathbb{I}$, we use lower indices. For example, $t=(t_1, t_2)$ for $t\in\mathbb{I}$. We sometimes write $t=(t_i, t_{\bar{i}})$ regardless of whether $i=1$ or 2, where $\bar{i}$ denotes the complementary index $3-i$ of $i$. Given an interval $J$ in $\mathbb{I}$, $J^1=]s_1^J, t_1^J]\times[0, s_2^J]$ resp. $J^2=[0, s_1^J]\times]s_2^J, t_2^J]$ is the 1-shadow resp. 2-shadow of $J$. For a function $f: I\to\mathbb{R}$, the increment of $f$ over an interval $J$ in I will be written $\Delta_J f$. This also applies to functions $f:\mathbb{I}\to\mathbb{R}$. Here

$$\Delta_J f=f(t^J)-f(s_1^J, t_2^J)-f(t_1^J, s_2^J)+f(s^J).$$

$f$ is called increasing, if $\Delta_J f\geq 0$ for all intervals $J$, of bounded variation, if $\sum_{j\in\mathbb{J}}|\Delta_J f|$ is bounded as a function of the partitions $\mathbb{J}$ of the parameter interval, regular, if

$$\lim_{s\downarrow t} f(s)=f(t),\quad \lim_{s\uparrow t} f(s),\quad \lim_{s_1\uparrow t_1, s_2\downarrow t_2} f(s),\quad \lim_{s_1\downarrow t_1, s_2\uparrow t_2} f(s)$$

exist, for all $t\in\mathbb{I}$.

Our basic probability space is $(\Omega, \mathcal{F}, P)$. $\mathcal{F}$ is assumed to be complete with respect to $P$. The filtration $\mathbb{F}=(\mathcal{F}_t)_{t\in\mathbb{I}}$ which is fixed throughout the paper is supposed to satisfy some basic assumptions: it is right continuous, i.e. $\mathcal{F}_t=\bigcap_{s>t}\mathcal{F}_s$, it is complete, i.e. $\mathcal{F}_t$ contains all $P$-zero sets, $t\in\mathbb{I}$, and, for convenience, $\mathcal{F}_t$ is trivial whenever $t\in\mathbb{I}\cap\partial\mathbb{R}_+^2$. The most important hypothesis, however, is the "conditional independence" of the filtrations $\mathbb{F}_1=(\mathcal{F}_{t_1}^1)_{t_1\in I}$ and $\mathbb{F}_2=(\mathcal{F}_{t_2}^2)_{t_2\in I}$, where $\mathcal{F}_{t_i}^i=\mathcal{F}_{(t_i,1)}$, $i=1,2$. It states that for all $t\in\mathbb{I}$, the $\sigma$-algebras $\mathcal{F}_{t_1}^1$ and $\mathcal{F}_{t_2}^2$ are conditionally independent given $\mathcal{F}_t$, and is often referred to as the $(F4)$-condition of Cairoli and Walsh [5].

Stochastic processes are a priori no more than families of random variables. A stochastic process $X$ on $\Omega\times\mathbb{I}$ defines two families of one-parameter processes: for $t_{\bar{i}}\in$ I, $X_{(\cdot,t_{\bar{i}})}$ is the process $(\omega, t_i)\to X_t(\omega)$, $i=1,2$. A process $X$ is called increasing resp. of bounded variation resp. regular if for all $\omega\in\Omega$ the function $X(\omega,\cdot)$ has the respective property. Two processes $X$, $Y$ are considered as being versions of each other if $X_t=Y_t$ a.s. for all $t$. Among the topologies on the subspaces of the linear space of $\mathcal{F}$-measurable random variables we will have to deal with the following: on the one hand, the usual $L^p$-topologies which are generated by the functionals $\|\xi\|_p=E(|\xi|^p)^{1/p}$ resp. $E(|\xi|^p)$ resp. $E(|\xi|\wedge 1)$ for $p\geq 1$ resp. $0<p<1$ resp. $p=0$; the corresponding topological vector spaces are denoted by $L^p(\Omega, \mathcal{F}, P)$; on the other hand, the space of all random variables $\xi$ for which $E(|\xi|\log^+|\xi|)<\infty$; this space which is topologized by $\|\xi\|_{L\log^+ L}=\inf\{\lambda>0: E(|\xi|/\lambda\log^+|\xi|/\lambda)\leq 1\}$, is denoted by $L\log^+ L$.

The $\sigma$-algebra $\mathbb{G}$ of optional sets in $\Omega\times\mathbb{I}$ is generated by the regular $\mathbb{F}$-adapted processes. To analyze the jumps of processes on $\Omega\times\mathbb{I}$, the following concept of "thin" optional sets proves to be useful. A set $T\in\mathbb{G}$ is called 0-simple if $\omega\to|T_\omega|$ is integrable.

The most important class of processes to be discussed here are the martingales and their quadratic variations. An integrable, $\mathbb{F}$-adapted process $M$ on $\Omega \times \mathbb{I}$ is called martingale if for $s, t \in \mathbb{I}$, $s \leq t$, we have $E(M_t \mid \mathcal{F}_s) = M_s$. A martingale $M$ is said to be $L \log^+ L$-integrable (-bounded) resp. $L^p$-integrable (-bounded) if $M_1 \in L \log^+ L$ resp. $L^p(\Omega, \mathcal{F}, P)$, $p \geq 1$. According to the regularity theorem of Bakry, Millet and Sucheston [2, 10] any $L \log^+ L$-integrable martingale possesses a version with regular trajectories. For regular martingales, the following three kinds of jumps are well defined and relevant. A point $(\omega, t) \in \Omega \times \mathbb{I}$ is called 0-jump if

$$\Delta_t M(\omega) = \lim_{s \uparrow t} \Delta_{]s,t]} M(\omega) \neq 0,$$

$i$-jump if

$$\Delta_t M(\omega) = 0 \quad \text{and} \quad \Delta_{t_i} M_{(\cdot, t_{\bar{i}})}(\omega) = \lim_{s_i \uparrow t_i} \Delta_{]s_i, t_i]} M_{(\cdot, t_{\bar{i}})}(\omega) \neq 0, \quad i = 1, 2.$$

In [9, p. 120], it is shown that the set of 0-jumps of a regular martingale is contained in a countable union of 0-simple sets. If, moreover, the martingale $M$ is $L^2$-integrable (square integrable), it can be decomposed by three jump parts $M^0$, $M^1$, $M^2$ consisting of the compensated jumps of $M$ of the respective kinds and a continuous part $M^c$ (see [9, p. 156]). The most general existence theorem for quadratic variation (see [9, p. 161]) states that any $L \log^+ L$-integrable martingale $M$ possesses a quadratic variation $[M]$. For any $t \in \mathbb{I}$,

$$[M]_t = (L^0\text{-}) \lim_{m \to \infty} \sum_{J \in \mathbb{J}_m} (\Delta_{J \cap [0,t]} M)^2$$

along any 0-sequence $(\mathbb{J}_m)_{m \in \mathbb{N}}$ of partitions of $\mathbb{I}$. If $M$ is $L^2$-integrable in addition, $[M]$ is simply the sum of the quadratic variations of its four components (see [9, pp. 159, 160]). By $[M]^i_{(\cdot, t_{\bar{i}})}$ we denote the quadratic variation of the one-parameter martingale $M_{(\cdot, t_{\bar{i}})}$, $t_{\bar{i}} \in \mathrm{I}$, $i = 1, 2$. Two processes $X$ and $Y$ are said to have orthogonal variation if for any 0-sequence $(\mathbb{J}_m)_{m \in \mathbb{N}}$ of partitions of the parameter interval $\sum_{J \in \mathbb{J}_m} |\Delta_J X \Delta_J Y| \to 0$ in probability as $m \to \infty$. We finally emphasize that, for convenience of notation, all martingales to be considered are assumed to vanish on $\mathbb{I} \cap \partial \mathbb{R}^2_+$.

## 1. Some convergence properties of quadratic variation

In this section of predominantly auxiliary character we will be occupied with the following problems concerning a given sequence $(M^n)_{n \in \mathbb{N}}$ of regular $L \log^+ L$-bounded martingales. Firstly, assume that for some regular $L \log^+ L$-integrable martingale $M$ and some $p \geq 0$, the sequence $([M^n - M]_1)_{n \in \mathbb{N}}$ of quadratic variations converges in $L^p(\Omega, \mathcal{F}, P)$ to 0, we deduce that $([M^n])_{n \in \mathbb{N}}$ converges to $[M]$ uniformly in $L^p(\Omega, \mathcal{F}, P)$. An analogous result for the quadratic $i$-variations $[M]^i$, $i = 1, 2$, will follow. Next, we will consider the continuity of the property of having orthogonal variation. We suppose that $(M^n)_{n\in\mathbb{N}}$ converges to $M$ in $L\log^+ L$ and that with respect to some $L\log^+ L$-bounded regular martingale $Q$ all $M^n$ have orthogonal variation. Then this property transfers to the pair $(M, Q)$. Our last problem is connected with the relationship between quadratic variations and quadratic $i$-variations of two-parameter martingales. We will investigate in which way the convergence of $([M^n]^i)_{n\in\mathbb{N}}$ and $([M^n])_{n\in\mathbb{N}}$ are related.

**Proposition 1.** *Let $p\geq 0$, $M^n$, $n\in\mathbb{N}$, $M$ be regular martingales in $L\log^+ L$ such that $[M]_1\in L^p(\Omega,\mathcal{F},P)$ and $[M^n-M]_1\to 0$ in $L^p(\Omega,\mathcal{F},P)$. Then $([M^n])_{n\in\mathbb{N}}$ converges uniformly to $[M]$ in $L^p(\Omega,\mathcal{F},P)$.*

**Proof.** Let $(\mathbb{J}_m)_{n\in\mathbb{N}}$ be an arbitrary 0-sequence of partitions of $\mathbb{I}$. Then for any $t\in\mathbb{I}$, $n\in\mathbb{N}$,

$$\begin{aligned}
|[M^n]_t-[M]_t| &= \lim_{m\to\infty}\left|\sum_{J\in\mathbb{J}_m}\{(\Delta_{J\cap[0,t]}M^n)^2-(\Delta_{J\cap[0,t]}M)^2\}\right| \\
&= \lim_{m\to\infty}\left|\sum_{J\in\mathbb{J}_m}\{\Delta_{J\cap[0,t]}(M^n-M)\Delta_{J\cap[0,t]}(M^n+M)\}\right| \\
&\leq \lim_{m\to\infty}\left\{\sum_{J\in\mathbb{J}_m}(\Delta_{J\cap[0,t]}(M^n-M))^2\sum_{J\in\mathbb{J}_m}(\Delta_{J\cap[0,t]}(M^n+M))^2\right\}^{1/2} \\
&= \{[M^n-M][M^n+M]_t\}^{1/2} \\
&\leq \{[M^n-M][M^n+M]_1\}^{1/2}.
\end{aligned}$$

Therefore, for any $n\in\mathbb{N}$,

$$\sup_{t\in\mathbb{I}}|[M^n]_t-[M]_t|\leq\{[M^n-M]_1[M^n+M]_1\}^{1/2}. \tag{1}$$

Moreover, note that $M\to([M]_1)^{1/2}$ is subadditive on $L\log^+ L$. Hence, due to the convergence of $([M^n-M]_1)_{n\in\mathbb{N}}$, the sequences $([M^n]_1)_{n\in\mathbb{N}}$ and consequently $([M^n+M]_1)_{n\in\mathbb{N}}$ are bounded in $L^p(\Omega,\mathcal{F},P)$. Now (1) combined with the inequality of Cauchy-Schwarz yields the desired conclusion. □

For the quadratic $i$-variations we have the following result.

**Proposition 2.** *Let $p\geq 0$, $M^n$, $n\in\mathbb{N}$, $M$ be regular martingales in $L\log^+ L$ such that $\sup_{t_2\in I}[M]^1_{(1,t_2)}\in L^p(\Omega,\mathcal{F},P)$ and $\sup_{t_2\in I}[M^n-M]^1_{(1,t_2)}\to 0$ in $L^p(\Omega,\mathcal{F},P)$. Then $([M^n]^1)_{n\in\mathbb{N}}$ converges uniformly to $[M]^1$ in $L^p(\Omega,\mathcal{F},P)$. An analogous statement holds for the quadratic 2-variations.*

**Proof.** In the same way as in the preceding proof, one deduces

$$\sup_{t\in\mathbb{I}}|[M^n]^1_t-[M]^1_t|\leq\left\{\sup_{t_2\in I}[M^n-M]^1_{(1,t_2)}\sup_{t_2\in I}[M^n+M]^1_{(1,t_2)}\right\}^{1/2} \tag{2}$$

and uses this to complete the proof. □

As a consequence of Proposition 1, we can sharpen the convergence result of [9, p. 136].

**Corollary.** *Let $(M^n)_{n\in\mathbb{N}}$ be a sequence of regular martingales converging in $L\log^+ L$ to a regular martingale M. Then their quadratic variations resp. quadratic i-variations converge uniformly to the quadratic variation resp. quadratic i-variation of M in $L^p(\Omega, \mathcal{F}, P)$ for $0\le p<\frac{1}{2}$, $i=1,2$.*

**Proof.** By [9, p. 163, Theorem 1, (iii)] we have for $\lambda>0$, $n\in\mathbb{N}$

$$\lambda P(\{[M^n-M]_1\}^{1/2}>\lambda)\le c_1\|M_1^n-M_1\|_{L\log^+ L}. \tag{3}$$

Also, by [9, p. 133, Proposition 1] and by Davis' inequality we have, for $\lambda>0$, $n\in\mathbb{N}$,

$$\begin{aligned}\lambda P\Big(\sup_{t_2\in \mathrm{I}}\{[M^n-M]^1_{(1,t_2)}\}^{1/2}>\lambda\Big)&\le E(\{[M^n-M]^1_1\}^{1/2})\\ &\le c_2 E\Big(\sup_{t_1\in\mathrm{I}}|M^n_{(t_1,1)}-M_{(t_1,1)}|\Big)\\ &\le c_3\|M_1^n-M_1\|_{L\log^+ L}.\end{aligned} \tag{4}$$

Here $c_1$, $c_2$, $c_3$ are constants which do not depend on $M^n$, $n\in\mathbb{N}$, $M$. Now suppose that the nonnegative random variable $\xi$ satisfies for some constant $c$ the inequalities

$$\lambda P(\xi>\lambda)\le c,\quad \lambda>0.$$

Then the elementary equation $E(\xi^p)=\int_{[0,\infty[}P(\xi^p>\lambda)\,\mathrm{d}\lambda$ implies

$$E(\xi^p)\le 1+c\int_{[1,\infty[}\lambda^{-1/p}\,\mathrm{d}\lambda<\infty\quad \text{for } p<1.$$

Therefore, on the one hand, (3) and (4) imply the boundedness of the sequences $([M^n-M]_1)_{n\in\mathbb{N}}$ and $(\sup_{t_2\in\mathrm{I}}[M^n-M]^1_{(1,t_2)})_{n\in\mathbb{N}}$ in $L^p(\Omega,\mathcal{F},P)$ for $0<p<\frac{1}{2}$. But on the other hand, they entail that the two sequences converge to 0 in $L^0(\Omega,\mathcal{F},P)$. Now an appeal to Vitali's classical convergence theorem all ,ws to deduce the hypotheses of Proposition 1 resp. Proposition 2. This completes the proof. □

The following proposition on the continuity of the property of having orthogonal variation will be stated both for one- and two-parameter martingales.

**Proposition 3.** 1. *Let $M^n$, $n\in\mathbb{N}$, M, Q be regular martingales, indexed by I. Assume $M_1^n\to M_1$ in $L^1(\Omega,\mathcal{F},P)$ and that $M^n$ and Q have orthogonal variation for all $n\in\mathbb{N}$. Then M and Q have orthogonal variation.*

2. *Let $M^n$, $n\in\mathbb{N}$, M, Q be regular martingales in $L\log^+ L$. Assume $M_1^n\to M_1$ in $L\log^+ L$ and that $M^n$ and Q have orthogonal variation for all $n\in\mathbb{N}$. Then M and Q have orthogonal variation.*

**Proof.** We argue for the two-parameter case. Let $(\mathbb{J}_m)_{m\in\mathbb{N}}$ be a 0-sequence of partitions of $\mathbb{I}$. By employing a diagonal sequence argument if necessary, we may assume that for all $n\in\mathbb{N}$ we have (a.s.)

$$[M^n-M]_1=\lim_{m\to\infty}\sum_{J\in\mathbb{J}_m}(\Delta_J(M^n-M))^2,$$

$$[Q]_1=\lim_{m\to\infty}\sum_{J\in\mathbb{J}_m}(\Delta_J Q)^2,\quad \lim_{m\to\infty}\sum_{J\in\mathbb{J}_m}|\Delta_J M^n\Delta_J Q|=0.$$

Then, for any $n\in\mathbb{N}$,

$$\begin{aligned}&\limsup_{m\to\infty}\sum_{J\in\mathbb{J}_m}|\Delta_J M\Delta_J Q|\\ &\quad\le\limsup_{m\to\infty}\sum_{J\in\mathbb{J}_m}|\Delta_J(M-M^n)\Delta_J Q|+\lim_{m\to\infty}\sum_{J\in\mathbb{J}_m}|\Delta_J M^n\Delta_J Q|\\ &\quad\le\limsup_{m\to\infty}\left\{\sum_{J\in\mathbb{J}_m}(\Delta_J(M-M^n))^2\sum_{J\in\mathbb{J}_m}(\Delta_J Q)^2\right\}^{1/2}\\ &\quad=\{[M^n-M][Q]_1\}^{1/2}.\end{aligned}$$

Since $n$ is arbitrary and since $[M^n-M]_1\to 0$ in $L^0(\Omega,\mathcal{F},P)$ by (3), the assertion readily follows. □

The quadratic $i$-variation and the quadratic variation of martingales which are continuous in the $\bar{i}$-direction will be linked by the subsequent proposition.

**Proposition 4.** *For every $0<p<\frac{1}{2}$ there exists a constant $c_p$ such that for any regular martingale $M$ in $L\log^+ L$ which has at most 1-jumps*

$$E(|[M]_1^1-[M]_1|^p)\le c_p E\left(\left\{\sup_{t_2\in\mathbb{I}}[M]^1_{(1,t_2)}[M]_1\right\}^{p/2}\right).$$

A similar statement holds for $[M]^2$.

**Proof.** Let $(\mathbb{J}_m)_{m\in\mathbb{N}}$ be an arbitrary sequence of partitions of $\mathbb{I}$. Observe first that by definition of the quadratic variations

$$[M]_1^1-[M]_1=2\lim_{m\to\infty}\sum_{J\in\mathbb{J}_m}\Delta_{J^1}M\Delta_J M\quad\text{in } L^0(\Omega,\mathcal{F},P).\tag{5}$$

For $m$, $l\in\mathbb{N}$ let

$$N^{m,l}_{\cdot}=\sum_{J\in\mathbb{J}_m}(-l)\wedge(\Delta_{J^1}M\vee l)\Delta_{J\cap[0,(1,\cdot)]}M,$$

$$\xi_m=\sup_{J_2\in(\mathbb{J}_m)_2}\sum_{J_1\in(\mathbb{J}_m)_1}(\Delta_{J_1}M_{(\cdot,s_2^J)})^2,$$

$$\eta_m=\sum_{J\in\mathbb{J}_m}\Delta_{J_2}[\Delta_{J_1}M_{(\cdot,\cdot)}]^2.$$

Here and in the following arguments $J_i$ denotes the projection of an interval $J$ on the $i$-axis, $\mathbb{J}_i$ the collection of all $J_i$ for $J \in \mathbb{J}$, if $\mathbb{J}$ is a partition of $\mathbb{I}$. Now fix $m, l \in \mathbb{N}$. Abbreviate $\mathbb{K}_n = (\mathbb{J}_n)_2$, $n \in \mathbb{N}$. Note that, since $M$ has at most 1-jumps, $N^{m,l}$ is a continuous one-parameter martingale with respect to $\mathbb{F}_2$. Its quadratic variation $[N^{m,l}]^2$ satisfies the following inequality

$$[N^{m,l}]_1^2 = \lim_{n\to\infty} \sum_{K\in\mathbb{K}_n} (\Delta_K N^{m,l})^2$$

$$= \lim_{n\to\infty} \sum_{J_2\in(\mathbb{J}_m)_2} \sum_{J_2\supset K\in\mathbb{K}_n} \left( \sum_{J_1\in(\mathbb{J}_m)_1} (-l)\wedge(\Delta_{J^1}M \vee l)\Delta_{J_1\times K}M \right)^2$$

$$\leq \sum_{J_2\in(\mathbb{J}_m)_2} \left\{ \sum_{J_1\in(\mathbb{J}_m)_1} (\Delta_{J^1}M)^2 \lim_{n\to\infty} \sum_{J_2\supset K\in\mathbb{K}_n} \sum_{J_1\in(\mathbb{J}_m)_1} (\Delta_{J_1\times K}M)^2 \right\}$$

(Cauchy–Schwarz)

$$\leq \xi_m \eta_m. \tag{6}$$

As in the proof of the above corollary, there are constants $b_1$, $b_2$ such that for all $m \in \mathbb{N}$, $\lambda > 0$

$$\lambda P(\eta_m^{1/2} > \lambda) \leq b_1 \|M\|_{L\log^+ L} \quad \text{(see [9, p. 76, Theorem 2(iii)])}, \tag{7}$$

$$\lambda P(\xi_m^{1/2} > \lambda) \leq b_2 \|M\|_{L\log^+ L}. \tag{8}$$

Just as (3) and (4), (7) and (8) imply

$$\sup_{m\in\mathbb{N}} E(\eta_m^p) < \infty, \tag{9}$$

$$\sup_{m\in\mathbb{N}} E(\xi_m^p) < \infty \quad \text{for } 0 < p < \tfrac{1}{2}. \tag{10}$$

(9) and (10) in turn, assisted by the inequality of Cauchy–Schwarz, entail that

$$\{(\xi_m \eta_m)^{1/2} : m \in \mathbb{N}\} \quad \text{is uniformly } p\text{-integrable for } 0 < p < \tfrac{1}{2}. \tag{11}$$

Next, we apply the inequality of Burkholder, Davis and Gundy (see [1, p. 14]) and (6) to the continuous $\mathbb{F}_2$-martingale $N^{m,l}$ to obtain for any $0 < p < 1$ a constant $a_p$ independent of $M$ such that

$$E(|N^{m,l}|^p) \leq a_p E(\{[N^{m,l}]_1^2\}^{p/2}) \leq a_p E(\{\xi_m \eta_m\}^{p/2}).$$

It remains to pass to the limits on both sides of (12) which is justified by (11), and to remember (5). This completes the proof. □

The following proposition allows to trace back the convergence of the quadratic variations of a sequence of martingales to a corresponding statement for the quadratic $i$-variations and will be very useful in the next section.

**Proposition 5.** *Let $(M^n)_{n\in\mathbb{N}}$ be a sequence of regular martingales in $L\log^+ L$ which have at most 1-jumps. Assume that the sequences $([M^n]_1)_{n\in\mathbb{N}}$ and $(\sup_{t_2\in\mathrm{I}}[M^n]^1_{(1,t_2)})_{n\in\mathbb{N}}$ are bounded in $L^p(\Omega,\mathcal{F},P)$ for $0\le p<\frac{1}{2}$. Then: if $(\sup_{t_2\in\mathrm{I}}[M^n]^1_{(1,t_2)})_{n\in\mathbb{N}}$ converges to 0 in $L^0(\Omega,\mathcal{F},P)$, $([M^n]_1)_{n\in\mathbb{N}}$ converges to 0 in $L^p(\Omega,\mathcal{F},P)$ for $0\le p<\frac{1}{2}$. A similar statement holds with respect to 2-jumps and quadratic 2-variations.*

**Proof.** Proposition 4 and the inequality of Cauchy–Schwarz imply that the sequence $([M^n]^1_1-[M^n]_1)_{n\in\mathbb{N}}$ converges to 0 in $L^p(\Omega,\mathcal{F},P)$ for $0\le p<\frac{1}{2}$. To finish the proof, one only has to remark that

$$[M^n]^1_1\le\sup_{t_2\in\mathrm{I}}[M^n]^1_{(1,t_2)},\quad n\in\mathbb{N},$$

and that, by Vitali's convergence theorem, $(\sup_{t_2\in\mathrm{I}}[M^n]^1_{(1,t_2)})_{n\in\mathbb{N}}$ goes to 0 in $L^p(\Omega,\mathcal{F},P)$ as $n\to\infty$ for $0\le p<\frac{1}{2}$. □

## 2. The continuity of the quadratic variation

In this section, we will use the convergence results just obtained to derive the continuity properties of the quadratic variation $[M]$ of a regular martingale $M$ in $L\log^+ L$. Our principal method can be outlined as follows. Consider a sequence $(M^n)_{n\in\mathbb{N}}$ of "small" martingales approximating the "large" martingale $M$. Here "small" means "bounded" or, if this cannot be achieved, at least having better integrability properties. Due to the smallness of the $M^n$, we possess information on the continuity properties of the quadratic variations $[M^n]$. We make use of this information and apply Propositions (1.1) and (1.2) to deduce results on the continuity of $[M]$.

If $M$ has no 0-jumps, we approximate it by its "truncations" $M^n=E((-n)\wedge(M_1\vee n)\,|\,\mathcal{F}\cdot)$, $n\in\mathbb{N}$. Of course, $M^n$ may have 0-jumps. But proposition (1.3) will enable us to see that the 0-jump parts $(M^n)^0$ and $M$ have orthogonal variation. This will imply that $[M^n-(M^n)^0-M]_1\to 0$, so that proposition (1.1) is applicable.

In case for $i=1$ or 2, $M$ has at most $i$-jumps, it can be approximated by a sequence of bounded martingales which have the same continuity properties. Then, the corollary of Propositions (1.1) and (1.2) gives the desired result.

If $M$ is continuous, we encounter the most difficult problem. As in the preceding case, we can approximate by a sequence $(M^n)_{n\in\mathbb{N}}$ of bounded martingales with at most 1-jumps (or at most 2-jumps). Proposition (1.3) allows us to conclude that the jump parts $(M^n_{(\cdot,t)})^0$ of $M^n_{(\cdot,t)}$ and $M_{(\cdot,t)}$ have orthogonal variation for $t\in\mathrm{I}$. As a consequence of this, $[M^n_{(\cdot,t)}-(M^n_{(\cdot,t)})^0-M_{(\cdot,t)}]^1_1\to 0$. Now Proposition (1.5) forces $[M^n-(M^n)^1-M]_1\to 0$, so that Proposition (1.1) is again applicable.

The methods just sketched, however, could not be seen to yield corresponding results for the remaining possible combinations of different kinds of jumps.

For the definition of the optional projections in the two parameter directions of bounded processes which appear in the proposition below, see for example [9, p. 49].

**Proposition 1.** 1. *Let $M$ be a regular martingale in $L\log^+ L$. Then the sequence of bounded martingales*

$$M^n = E((-n)\wedge(M_1 \vee n)\,|\,\mathcal{F}_\cdot),\quad n\in\mathbb{N},$$

*converges to $M$ in $L\log^+ L$.*

2. *Let $M$ be a regular martingale in $L\log^+ L$ which has at most 1-jumps. For $n\in\mathbb{N}$, let $N^n$ be the continuous one-parameter martingale with respect to $\mathbb{F}_2$ defined by stopping with respect to $\mathbb{F}_2$ the continuous martingale $M_{(1,\cdot)}$ when its absolute value exceeds $n$ for the first time, $M^n$ the optional projection of $N^n$ in direction 1. Then the bounded martingales $M^n$, $n\in\mathbb{N}$, have at most 1-jumps and converge to $M$ in $L\log^+ L$. A similar statement holds with respect to the second parameter.*

**Proof.** 1. The first part follows from the simple fact that

$$|M_1^n - M_1| \le |M_1| 1_{\{|M_1|>n\}},\quad n\in\mathbb{N}.$$

2. Let us turn to the less trivial second assertion. By [9, p. 62], Theorem 1, $M^n$ possesses at most 1-jumps. Moreover, by definition, $M_1^n \to M_1$ a.s. To complete the proof, we therefore have to show that the sequence $(|M_1^n|\log^+|M_1^n|)_{n\in\mathbb{N}}$ is uniformly integrable. To this end, according to Burkholder, Davis and Gundy [3, p. 238], we may choose a convex function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$\phi(0)=0,\quad \lim_{t\to\infty}\phi(t)/t=\infty \quad\text{and}\quad E(\phi(|M_1|\log^+|M_1|))<\infty.$$

Since the process $\phi(|M_{(1,\cdot)}|\log^+|M_{(1,\cdot)}|)$ is a submartingale with respect to $\mathbb{F}_2$, we have

$$\begin{aligned}&\sup_{n\in\mathbb{N}} E(\phi(|M_1^n|\log^+|M_1^n|))\\ &\quad\le E(\phi(|M_1|\log^+|M_1|)) + \sup_{n\in\mathbb{N}} E(\phi(n\log^+ n)1_{\{\sup_{t_2\in I}|M_{(1,t_2)}|>n\}})\\ &\quad\le 2E(\phi(|M_1|\log^+|M_1|))<\infty.\end{aligned}$$

Now we only have to apply the lemma of de la Vallée-Poussin to obtain the desired uniform integrability. □

In the next step we will show that, if $M$ has no 0-jumps, then the 0-jump parts $(M^n)^0$ of the approximations $M^n$ according to the first part of Proposition 1, and $M^n-(M^n)^0-M$ have orthogonal variation. Similarly, if $M$ is continuous, the jump parts $(M^n_{(\cdot,t)})^0$ of the approximations according to the second part of Proposition 1, and $M^n_{(\cdot,t)}-(M^n_{(\cdot,t)})^0-M_{(\cdot,t)}$ have orthogonal variation for all $t\in I$.

**Proposition 2.** 1. *Let $M$ be a regular martingale in $L\log^+ L$ without 0-jumps, $M^n$, $n \in \mathbb{N}$, according to Proposition 1,1. Then $(M^n)^0$ and $M^n-(M^n)^0-M$ have orthogonal variation for all $n \in \mathbb{N}$.*

2. *Let $M$ be a continuous martingale in $L\log^+ L$, $M^n$, $n \in \mathbb{N}$, according to Proposition 1,2. Then $(M^n_{(\cdot,t)})^0$ and $M^n_{(\cdot,t)}-(M^n_{(\cdot,t)})^0-M_{(\cdot,t)}$ have orthogonal variation for all $n \in \mathbb{N}$, $t \in \mathrm{I}$.*

**Proof.** We argue for 1, the proof of 2 being parallel. Fix $n \in \mathbb{N}$. By [9, p. 152], $(M^n)^0$ can be approximated in $L^p(\Omega, \mathcal{F}, P)$ for all $p \geq 0$ by compensated jumps of $M^n$ on 0-simple sets. Hence Proposition (1.3), 2 tells us that we may assume $(M^n)^0$ to be the compensated jump of $M^n$ on a fixed 0-simple set. Let $(\mathbb{J}_m)_{m\in\mathbb{N}}$ be a 0-sequence of partitions of $\mathbb{I}$. Since $(M^n)^0$ is of integrable variation (see [9, p. 126]), we can associate with it a random measure $\mu$ which describes its variation and is a.s. finite. For $m \in \mathbb{N}$, let

$$Y_m = \sum_{J\in\mathbb{J}_m} \Delta_J(M^n-(M^n)^0-M)1_J.$$

Then $Y_m \to 0$ as $m \to \infty$ a.s., since $M^n-(M^n)^0-M$ has no 0-jumps. This convergence is dominated by the finite random variable $4\sup_{t\in\mathbb{I}}(|M^n_t|+|(M^n)^0_t|+|M_t|)$. Therefore

$$\sum_{J\in\mathbb{J}_m} |\Delta_J(M^n-(M^n)^0-M)\Delta_J(M^n)^0| \leq \int_{\mathbb{I}} |Y_m|\,\mathrm{d}\mu \to 0 \quad \text{a.s.}$$

as $m \to \infty$, hence in $L^0(\Omega, \mathcal{F}, P)$. This completes the proof. $\square$

We are prepared to state our first main result.

**Theorem 1.** *Let $M$ be a regular martingale in $L\log^+ L$ without 0-jumps. Then $[M]$ has no 0-jumps.*

**Proof.** Let $M^n$, $n \in \mathbb{N}$, be given by Proposition 1,1. Then (3) implies $[M^n-M]_1 \to 0$ in $L^0(\Omega, \mathcal{F}, P)$. Moreover, by Proposition 2,1, for all $n \in \mathbb{N}$,

$$[M^n-M]_1 = [M^n-(M^n)^0-M]_1 + [(M^n)^0]_1.$$

Therefore $[M^n-(M^n)^0-M]_1 \to 0$ in $L^0(\Omega, \mathcal{F}, P)$. Hence $[M^n-(M^n)^0] \to [M]$ uniformly in $L^0(\Omega, \mathcal{F}, P)$ by Proposition (1.1). But, according to [9, p. 160], $[M^n-(M^n)^0]$ has no 0-jumps. Consequently $[M]$ has no 0-jumps. $\square$

If $M$ has no 0-jumps and no $i$-jumps for $i=1$ or 2, the approximation of the second part of Proposition 1 allows an easy conclusion.

**Theorem 2.** *Let $M$ be a regular martingale in $L\log^+ L$ which possesses at most 1-jumps. Then $[M]$ possesses at most 1-jumps. An analogous result holds with respect to 2-jumps.*

**Proof.** Let $M^n$, $n\in\mathbb{N}$, be the approximating sequence of $M$ according to Proposition 1,2. Again by (3), $[M^n-M]_1\to 0$ in $L^0(\Omega,\mathcal{F},P)$, and by Proposition (1.1), $[M^n]\to[M]$ uniformly in $L^0(\Omega,\mathcal{F},P)$. But $[M^n]$ has at most 1-jumps for all $n\in\mathbb{N}$, hence so does $[M]$. □

We finally turn to continuous $M$. Before we present our final main result, however, a few remarks concerning the methods of proof are in order.

**Remarks.** Given the procedure which yields Theorem 1, one is tempted to try the following method in order to deduce the continuity of $[M]$ from the continuity of $M$. (a) Approximate $M$ for example by the sequence $(M^n)_{n\in\mathbb{N}}$ of bounded martingales with at most 1-jumps according to Proposition 1,2. (b) Show that the 1-jump parts $(M^n)^1$ of $M^n$ and $M^n-(M^n)^1-M$ have orthogonal variation. (c) Conclude using Proposition (1.1). The difficult part of this seemingly natural procedure is to verify (b). To do this, it would be sufficient to see that $(M^n)^1$ and $M$ have orthogonal variation. But on the basis of the available Burkholder-Davis-Gundy type inequalities linking the quadratic variation of a two-parameter martingale to the supremum of its modulus, we were only able to prove orthogonality under more restrictive integrability assumptions than $L\log^+ L$. It seems conceivable to start with a Burkholder-Davis-Gundy type inequality for the $p$-norms, $0<p<1$, for continuous martingales. Such an inequality, however, although probably true, has not yet been rigorously proved (see [9, p. 165]) and is out of the scope of this paper. This drawback forced us to take resort to a procedure which relates the convergence of quadratic variations to the convergence of quadratic $i$-variations, as given by Proposition (1.5). It only requires to verify an analogon of (b) for the one-parameter "sections" of $M^n$ and $M$. Unfortunately, it produces another drawback: the question, whether $M$ inherits the remaining three combinations of different kinds of jumps to $[M]$ seems to be inaccessible by the methods involved.

**Theorem 3.** *Let $M$ be a continuous martingale in $L\log^+ L$. Then $[M]$ is continuous.*

**Proof.** Let $M^n$, $n\in\mathbb{N}$, be given by Proposition 1,2. The proof of the corollary of Propositions (1.1) and (1.2) shows that the sequences $([M^n-M]_1)_{n\in\mathbb{N}}$ and $(\sup_{t_2\in I}[M^n-M]^1_{(1,t_2)})_{n\in\mathbb{N}}$ converge to 0 in $L^p(\Omega,\mathcal{F},P)$ for $0\le p<\frac{1}{2}$. Moreover,

$$[M^n_{(\cdot,t_2)}-M_{(\cdot,t_2)}]^1_1=[M^n_{(\cdot,t_2)}-(M^n_{(\cdot,t_2)})^0-M_{(\cdot,t_2)}]^1_1+[(M^n_{(\cdot,t_2)})^0]^1_1,$$

for $n\in\mathbb{N}$, $t_2\in I$, due to Proposition 2,2. Therefore

$$\sup_{t_2\in I}[M^n_{(\cdot,t_2)}-(M^n_{(\cdot,t_2)})^0-M_{(\cdot,t_2)}]^1_1\to 0 \quad \text{in } L^p(\Omega,\mathcal{F},P) \text{ for } 0\le p<\tfrac{1}{2}.$$

Next,

$$[M^n]_1 = [(M^n)^1]_1 + [(M^n)^c]_1, \quad n \in \mathbb{N} \quad (\text{see } [9, \text{p. } 160]).$$

This implies that together with $([M^n]_1)_{n\in\mathbb{N}}$ also $([(M^n)^c]_1)_{n\in\mathbb{N}}$, and $([(M^n)^c - M]_1)_{n\in\mathbb{N}}$, is bounded in $L^p(\Omega, \mathcal{F}, P)$ for $0 \leq p < \frac{1}{2}$. Since, however, $(M^n_{(\cdot,t_2)})^0 = (M^n)^1_{(\cdot,t_2)}$ by definition for all $n \in \mathbb{N}$, Proposition (1.5) can finally be applied to yield

$$[(M^n)^c - M]_1 = [M^n - (M^n)^1 - M]_1 \to 0 \quad \text{in } L^p(\Omega, \mathcal{F}, P) \text{ for } 0 \leq p < \tfrac{1}{2}.$$

Proposition (1.1) now gives $[(M^n)^c] \to [M]$ uniformly in $L^p(\Omega, \mathcal{F}, P)$ for $0 \leq p < \frac{1}{2}$. Therefore $[M]$ is continuous along with $[(M^n)^c]$, $n \in \mathbb{N}$, by [9, p. 158]. □

As an easy consequence of Theorem 3 and its proof we note the following approximation property.

**Corollary.** *Let $M$ be a continuous martingale in $L \log^+ L$. Then there exists a sequence $(M^n)_{n\in\mathbb{N}}$ of continuous martingales which are q-integrable for all $q \geq 0$ and which converge uniformly in* 𝕀 *to $M$, in $L^p(\Omega, \mathcal{F}, P)$ for all $0 \leq p < 1$.*

**Proof.** For $n \in \mathbb{N}$, let $M^n$ be the continuous part of the $n$th approximation of $M$ according to Proposition 1,2. Then $M^n$ is $q$-integrable for all $q \geq 0$ by [9, p. 154]. The convergence assertion is then a consequence of the proof of Theorem 3 and of [9, p. 165, Theorem 2]. □

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