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A NEW DEFINITION OF RAIN INTENSITY BASED ON THE DISCONTINUOUS NATURE OF RAINFALL<br>Harry Pavlopoulos and Haim Kutiel<br>Department of Statistics<br>Athens University of Economics and Business<br>Technical Report No. 45, May 1998



## DEPARTMENT OF STATISTICS

# A New Definition of Rain Intensity based on the Discontinuous Nature of Rainfall 

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#### Abstract

The classical definition of rain rate as the quotient of height (or volume) of rainfall water, by the time elapsed during its accumulation in a measuring raingauge, is discussed. This definition implies or eludes to an inappropriate assumption of temporal continuity of rainfall. Our new definition is based on associating the population of raindrops produced during a rain event, with the corresponding statistical population of diameter sizes, and on the assumption that these random diameter sizes follow a discrete probability distribution possessing a finite third moment. Then, viewing the total rainfall accumulated in a raingauge as being the integral over time of the water volume added by each raindrop, and based on the above statistical assumption, it is shown that the volume $\mathrm{V}(\Delta \mathrm{t})$, the height $\mathrm{H}(\Delta \mathrm{t})$, and the classical measure of rain intensity R , corresponding to rainfall water accumulated during a time interval of given length $\Delta \mathrm{t}$, are all random integer multiples of certain fundamental quantities. We define rain rate as being the elapsed time $t_{0}$ between the arrivals of two consecutive raindrops of average size at the orifice of the measuring raingauge. According to this definition, it is shown that given a probability distribution of raindrop diameter sizes, the classical rain rate measured at the location of a raingauge of given orifice size $\alpha$, during a time interval of given length $\Delta t$, increases as $t_{0}$ decreases, and decays as $t_{0}$ increases. Moreover, this new approach may be applied towards evaluation of rainfall detectability by Radar, and possibly also in assessment of geomorphic implications of the erosional effect of rainfall.


KEY WORDS: Coefficient of Detectability, Discrete Probability Distribution, Rainfall Droplets, Rainfall Intensity, Temporal Discontinuity.
ABBREVIATED TITLE: Definition of Rain Intensity

## 1. INTRODUCTION

The occurrence of a rain event in a certain place at a certain time is often treated as being the outcome of a stochastic process, and the intensity of a rain event is also treated as a random variable. The intensity of a given rain event is expressed by a physical variable called rain rate, which is measured in units of millimetres per hour ( $\mathrm{mm} / \mathrm{hr}$ ), and reflects the height of the vertical column of accumulated rainfall per unit of time, at a fixed location where a measuring gauge has been installed. This is the classical definition of rain rate, which in fact quantifies the average intensity of rainfall over the area of the gauge's orifice during a finite time interval.

Specifically, let us agree on some small enough time step $\Delta t$ (say 1 hour or 1 minute or 1 second), and let us focus on the volume increment $\mathrm{V}(\Delta \mathrm{t})$ of the water accumulated in a rain-gauge of fixed orifice size $\alpha$ during the time step $\Delta \mathrm{t}$. Clearly,

$$
\begin{equation*}
\mathrm{V}(\Delta \mathrm{t})=\alpha \cdot \mathrm{H}(\Delta \mathrm{t}) \tag{1}
\end{equation*}
$$

where $\mathrm{H}(\Delta \mathrm{t})$ is the height increment of the cylindrical water column accumulated in the gauge during $\Delta t$, and the ratio

$$
\begin{equation*}
\mathrm{R}=\mathrm{H}(\Delta \mathrm{t}) / \Delta \mathrm{t}=\mathrm{V}(\Delta \mathrm{t}) /(\alpha \cdot \Delta \mathrm{t}) \tag{2}
\end{equation*}
$$

is the classical definition of average rain rate during a time interval of length $\Delta \mathrm{t}$, at the fixed location of the rain-gauge.

The above classical definition of rain rate subtly implies that, in fact, rain rate is a space and time average of some other, instantaneously (in time) and pointwise (in space) valued latent variable, which is impossible to measure and we shall refer to it as latent intensity. Another subtle implication of the classical definition of rain rate is that, in order for rain rate to express accurately the space-time average of the latent variable, this very latent variable must be varying continuously in space and in time.

However, it is well known that rain rate itself does not vary continuously over space, nor during time, from one value to another, and therefore, one should not expect that the latent intensity does so either. For example, it is common practice to use the quotient of total rainfall height by the length of a time interval during which the accumulation occurred, in order to calculate the (average) rain rate at a certain location. Clearly, if the time interval is long enough, say 24 hours, it may very well contain subintervals during which there was no rain. In such a case it is clear that the obtained value of rain rate shall be incorrect due to the temporal discontinuity of rainfall. Also the drawing of isohyetal maps of rainfall, based on interpolation techniques between data obtained from "pointwise" measurements of rainfall, relies heavily on the assumption of spatial continuity of rain fields. This assumption has been criticised widely by Kay and Kutiel (1994), and Kutiel and Kay (1996), who demonstrated the consequences of this inappropriate approach.

In order to "tame the demon" of temporal and spatial discontinuity of rain rate and/or of latent intensity, one may choose to consider only tiny scales of space and time, and to adopt that the assumptions of spatial and temporal continuity of rain rate hold true there. Still, we think that this would be unrealistic, because even in the finer scales of space and time, the mildest rainfall as well as the most intense storm materialise in the form of populations consisting of individual droplets of various
shapes and sizes which raid the orifice of the measuring gauge intermittently, and not as a continuous stream.

In this article we propose a new approach in order to quantify the notion of rainfall intensity. The proposed definition does not rely on any continuity assumptions regarding rainfall in space and/or in time. Instead, viewing rainfall as a statistical population of falling droplets whose size follows a discrete probability law, we argue in Section 2 that the classical notion of rain rate is indeed a quantized variable and not a continuous one. The proposed new definition of rain intensity relies on a statistical average of time between two "successive" arrivals of droplets into a measuring gauge, and is stated in Section 3. Section 4 relates the new definition of rain intensity to a coefficient of detectability of a rain droplet by radar. This relation suggests yet another possibility of defining rain intensity, linked to a different type of instrument used for measuring rainfall, and at the same time showing the inadequacy of radar for measuring rainfall. Section 5 concludes this article with some additional remarks.

## 2. ASSUMPTIONS ON A POPULATION OF RAIN DROPLETS

The accumulation of rainfall water in a rain-gauge, when it is raining, is certainly the result of addition of the volumes of numerous small droplets of water falling freely inside a vertical column of air just above the orifice of the gauge. These droplets are of many different sizes, and shapes, carrying different volumes of water, and therefore the velocities with which they fall also differ. Without much loss of generality, we may assume that, just before entering the measuring gauge, all the droplets have spherical shape, of various diameters though, and that they have reached terminal velocity. Empirical studies have shown that the terminal velocity of a rain droplet with diameter $\delta>0$ is given by the formula [Atlas and Ulbrich (1977), Sumner (1988)]:

$$
\begin{equation*}
\mathrm{U}(\delta)=\beta \cdot \delta^{\gamma} \tag{3}
\end{equation*}
$$

with empirically determined values for the parameters $\beta=3.78$, and $\gamma=0.67$. The volume of a spherical droplet with diameter $\delta>0$ is

$$
\begin{equation*}
\mathrm{V}_{\delta}=\pi \cdot \delta^{3} / 6 \tag{4}
\end{equation*}
$$

It is traditional in the meteorological literature to consider the range of sizes of rain droplets as being a continuum, and to model the probability distribution of the random drop size in terms of a probability density function defined on that range (e.g. gamma, log-normal, etc.); see Torres et al. (1994) and references therein. Our approach in this article deviates from that tradition. Instead, we shall assume that the size of a droplet's random diameter D attains only countably many possible values $\left\{\delta_{\mathrm{i}}\right.$ ; $\mathrm{i}=1,2, \ldots\}$, with corresponding probabilities

$$
\begin{equation*}
\mathrm{p}_{\mathrm{i}}=\mathrm{P}\left(\mathrm{D}=\delta_{i}\right) \geq 0 \quad \text { for } \mathrm{i}=1,2, \ldots, \quad \text { and } \quad \sum_{i=1}^{\infty} \mathrm{p}_{\mathrm{i}}=1 \tag{5}
\end{equation*}
$$

Let $N(\Delta t)$ denote the random total number of droplets which contributed to the increment $V(\Delta t)$ of the volume of the cylindrical column of water accumulated in the gauge, during a time interval of length $\Delta t$. Also, for each $i=1,2, \ldots$, let

$$
\mathrm{N}_{\mathrm{i}}(\Delta \mathrm{t})=\mathrm{p}_{\mathrm{i}} \cdot \mathrm{~N}(\Delta \mathrm{t})(6)
$$

denote the random expected number of droplets with diameter $\delta_{\mathrm{i}}$, conditionally on the value of $N(\Delta t)$, during the same time interval of length $\Delta t$, so that

$$
\begin{equation*}
\mathrm{N}(\Delta \mathrm{t})=\sum_{\mathrm{i}=1}^{\infty} \mathrm{N}_{\mathrm{i}}(\Delta \mathrm{t})=\mathrm{N}_{1}(\Delta \mathrm{t})+\mathrm{N}_{2}(\Delta \mathrm{t})+\cdots \tag{7}
\end{equation*}
$$

Moreover, for each $i=1,2, \ldots$, and conditionally on the value of $N(\Delta t)$, we may also define the random expected partial increment of the column's accumulated volume, due to the contribution by droplets of diameter $\delta_{i}$, during the time interval of length $\Delta t$

$$
\begin{equation*}
\mathrm{V}_{\mathrm{i}}(\Delta \mathrm{t})=\frac{\pi}{6} \cdot \delta_{\mathrm{i}}^{3} \cdot \mathrm{~N}_{\mathrm{i}}(\Delta \mathrm{t}) \tag{8}
\end{equation*}
$$

and analogously to equations (1) and (2) we define $H_{i}(\Delta t)$ and $R_{i}$ via the equations

$$
\begin{gather*}
\mathrm{V}_{\mathrm{i}}(\Delta \mathrm{t})=\alpha \cdot \mathrm{H}_{\mathrm{i}}(\Delta \mathrm{t})  \tag{9}\\
\mathrm{R}_{\mathrm{i}}=\mathrm{H}_{\mathrm{i}}(\Delta \mathrm{t}) / \Delta \mathrm{t}=\mathrm{V}_{\mathrm{i}}(\Delta \mathrm{t}) /(\alpha \cdot \Delta \mathrm{t}) \tag{10}
\end{gather*}
$$

That is, conditionally on the value of $\mathrm{N}(\Delta \mathrm{t}), \mathrm{H}_{\mathrm{i}}(\Delta \mathrm{t})=\mathrm{V}_{\mathrm{i}}(\Delta \mathrm{t}) / \alpha$ is the random expected partial increment of the column's height, and $\mathrm{R}_{\mathrm{i}}=\mathrm{H}_{\mathrm{i}}(\Delta \mathrm{t}) / \Delta \mathrm{t}$ is the random expected average rain rate, due to the contribution by droplets of diameter $\delta_{\mathrm{i}}$, during the time interval of length $\Delta t$.

### 2.1 Quantization of Classical Rain Rate

It is clear that if the third moment of the drop size distribution is finite:

$$
\begin{equation*}
\mathrm{E}\left(\mathrm{D}^{3}\right)=\sum_{\mathrm{i}=1}^{\infty} \mathrm{p}_{\mathrm{i}} \cdot \delta_{\mathrm{i}}^{3}<\infty \tag{11}
\end{equation*}
$$

then employing equations (6) and (8) we obtain the limit of the series

$$
\begin{equation*}
\sum_{\mathrm{i}=1}^{\infty} \mathrm{V}_{\mathrm{i}}(\Delta \mathrm{t})=\sum_{\mathrm{i}=1}^{\infty} \frac{\pi}{6} \cdot \delta_{\mathrm{i}}^{3} \cdot \mathrm{p}_{\mathrm{i}} \cdot \mathrm{~N}(\Delta \mathrm{t})=\frac{\pi}{6} \cdot \mathrm{E}\left(\mathrm{D}^{3}\right) \cdot \mathrm{N}(\Delta \mathrm{t}) \tag{12}
\end{equation*}
$$

Since the series (12) must balance with the total volume increment $V(\Delta t)$, we have

$$
\begin{equation*}
\mathrm{V}(\Delta \mathrm{t})=\sum_{\mathrm{i}=1}^{\infty} \mathrm{V}_{\mathrm{i}}(\Delta \mathrm{t})=\frac{\pi}{6} \cdot \mathrm{E}\left(\mathrm{D}^{3}\right) \cdot \mathrm{N}(\Delta \mathrm{t}) \tag{13}
\end{equation*}
$$

Similarly, under the assumption (11), using (6), (8), and (9) it follows that

$$
\begin{equation*}
\mathrm{H}(\Delta \mathrm{t})=\sum_{\mathrm{i}=1}^{\infty} \mathrm{H}_{\mathrm{i}}(\Delta \mathrm{t})=\frac{\pi}{6} \cdot \mathrm{E}\left(\mathrm{D}^{3}\right) \cdot \frac{\mathrm{N}(\Delta \mathrm{t})}{\alpha} \tag{14}
\end{equation*}
$$

and using (6), (8), and (10) we obtain that

$$
\begin{equation*}
\mathrm{R}=\sum_{\mathrm{i}=1}^{\infty} \mathrm{R}_{\mathrm{i}}=\frac{\pi}{6} \cdot \mathrm{E}\left(\mathrm{D}^{3}\right) \cdot \frac{\mathrm{N}(\Delta \mathrm{t})}{\alpha \Delta \mathrm{t}} \tag{15}
\end{equation*}
$$

As intermediate steps, equations (8), (9), and (10), combined with (6), yield

$$
\begin{align*}
& \mathrm{V}_{\mathrm{i}}(\Delta \mathrm{t})=\frac{\pi}{6} \cdot \delta_{\mathrm{i}}^{3} \cdot \mathrm{p}_{\mathrm{i}} \cdot \mathrm{~N}(\Delta \mathrm{t})  \tag{16}\\
& \mathrm{H}_{\mathrm{i}}(\Delta \mathrm{t})=\frac{\pi}{6 \alpha} \cdot \delta_{\mathrm{i}}^{3} \cdot \mathrm{p}_{\mathrm{i}} \cdot \mathrm{~N}(\Delta \mathrm{t})  \tag{17}\\
& \mathrm{R}_{\mathrm{i}}=\frac{\pi}{6 \alpha \Delta \mathrm{t}} \cdot \delta_{\mathrm{i}}^{3} \cdot \mathrm{p}_{\mathrm{i}} \cdot \mathrm{~N}(\Delta \mathrm{t}) \tag{18}
\end{align*}
$$

which combined with (13), (14), and (15), respectively, yield the analogies

$$
\begin{equation*}
\frac{\mathrm{V}_{\mathrm{i}}(\Delta \mathrm{t})}{\mathrm{V}(\Delta \mathrm{t})}=\frac{\mathrm{H}_{\mathrm{i}}(\Delta \mathrm{t})}{\mathrm{H}(\Delta \mathrm{t})}=\frac{\mathrm{R}_{\mathrm{i}}}{\mathrm{R}}=\frac{\mathrm{p}_{\mathrm{i}} \cdot \delta_{\mathrm{i}}^{3}}{\mathrm{E}\left(\mathrm{D}^{3}\right)} \tag{19}
\end{equation*}
$$

Equations (13), (14), and (15) show that under the assumption of a discrete drop size distribution, with finite third moment (i.e. $\mathrm{E}\left(\mathrm{D}^{3}\right)<\infty$ ), the increments of volume $\mathrm{V}(\Delta \mathrm{t})$, and of height $\mathrm{H}(\Delta \mathrm{t})$, of the water column accumulated in a rain-gauge of orifice size $\alpha$, during a time interval of length $\Delta t$, and also the average rain-rate during the same time interval at the location of the gauge, are random integer multiples of the "fundamental" quantities $\pi \mathrm{E}\left(\mathrm{D}^{3}\right) / 6, \quad \pi \mathrm{E}\left(\mathrm{D}^{3}\right) /(6 \alpha)$, and $\pi \mathrm{E}\left(\mathrm{D}^{3}\right) /(6 \alpha \Delta \mathrm{t})$ respectively.

The above argument demonstrates in an elementary manner the discontinuous nature of classical rain-rate in the course of time, at a fixed location of size comparable to that of a rain-gauge. The argument stems entirely from the single assumption that drop sizes do not form a continuum, but instead, drop diameter size is a random quantity with discrete probability distribution $\left\{p_{i} ; i=1,2, \ldots\right\}$, supported only on a countable set of possible values $\left\{\delta_{i} ; i=1,2, \ldots\right\}$.

## 3. A NEW DEFINITION OF RAIN INTENSITY

As we have already mentioned, $N(\Delta t)$ is the total (random) number of droplets that have been captured in the rain-gauge during a time interval of length $\Delta t>0$. The finite population of these $N(\Delta t)$ droplets consists of droplets with different sizes, and according to our previous model it is partitioned to countably many sub-populations,
each containing an expected number $N_{i}(\Delta t)=p_{i} \cdot N(\Delta t)$ of droplets with the same size of diameter $\delta_{i}$. Apparently, these droplets do not arrive one at a time into the gauge's catchment, but rather as layered or stratified groups of droplets, each group containing droplets of several sizes, distributed as a random spatial pattern of "points or disk-like sections’'- so to speak - over the gauge's orifice area [Guttorp (1995), pg. 251].

Despite this reality, let us idealise the formation of the droplets' arrivals, assuming that droplets arrive into the gauge as singletons and not as layered groups. Of course, each droplet may enter the gauge for landing by arriving anywhere in the area of the gauge's orifice. Moreover, let us assume that all the droplets have the same size, equal to the size of a fictitious prototype droplet whose volume equals the mean

$$
\begin{equation*}
V_{0}=\mathrm{E}\left(\frac{\pi}{6} \cdot \mathrm{D}^{3}\right)=\frac{\pi}{6} \cdot \mathrm{E}\left(\mathrm{D}^{3}\right)=\frac{\pi}{6} \cdot \sum_{\mathrm{i}=1}^{\infty} \mathrm{p}_{\mathrm{i}} \cdot \delta_{\mathrm{i}}^{3} \tag{20}
\end{equation*}
$$

of the probability distribution of the possible droplet volumes. Let us also assume that all the droplets are free-falling with fixed uniform terminal velocity, equal to the mean terminal velocity of the true droplet population. According to equation (3), this mean terminal velocity is

$$
\begin{equation*}
u_{0}=E(U)=E\left(\beta \cdot D^{\gamma}\right)=\beta \cdot E\left(D^{\gamma}\right)=\beta \cdot \sum_{i=1}^{\infty} p_{i} \cdot \delta_{i}^{\gamma} \tag{21}
\end{equation*}
$$

Under these idealisations, it is reasonable to consider an equipartition of the time interval during which the total of arrivals occurred, so that the interarrival time between any two consecutive droplets at the orifice of the measuring gauge is

$$
\begin{equation*}
\mathrm{t}_{0}=\frac{\Delta \mathrm{t}}{\mathrm{~N}(\Delta \mathrm{t})} \tag{22}
\end{equation*}
$$

It is this quantity $\mathrm{t}_{0}$ which we propose as a new definition of rain rate or intensity of a rain event of duration $\Delta \mathrm{t}$, consisting of a random total number of droplets $\mathrm{N}(\Delta \mathrm{t})$, whose diameters follow a discrete probability distribution with finite third moment.

It is important to note that $t_{0}$ is a random quantity, since $N(\Delta t)$ is random, assuming different values during the progression of a rain event, at any given location.

By solving each of the equations (13), (14), (15) for $\mathrm{N}(\Delta \mathrm{t})$, one obtains the following three equivalent expressions

$$
\begin{equation*}
\mathrm{N}(\Delta \mathrm{t})=\frac{6}{\pi \mathrm{E}\left(\mathrm{D}^{3}\right)} \cdot \mathrm{V}(\Delta \mathrm{t})=\frac{6 \alpha}{\pi \mathrm{E}\left(\mathrm{D}^{3}\right)} \cdot \mathrm{H}(\Delta \mathrm{t})=\frac{6 \alpha \Delta \mathrm{t}}{\pi \mathrm{E}\left(\mathrm{D}^{3}\right)} \cdot \mathrm{R} \tag{23}
\end{equation*}
$$

which if substituted into (22) yield three equivalent expressions of

$$
\begin{equation*}
t_{0}=\frac{\pi}{6} \mathrm{E}\left(\mathrm{D}^{3}\right) \cdot \frac{\Delta \mathrm{t}}{\mathrm{~V}(\Delta \mathrm{t})}=\frac{\pi}{6} \mathrm{E}\left(\mathrm{D}^{3}\right) \cdot \frac{\Delta \mathrm{t}}{\alpha \mathrm{H}(\Delta \mathrm{t})}=\frac{\pi}{6} \mathrm{E}\left(\mathrm{D}^{3}\right) \cdot \frac{1}{\alpha \mathrm{R}} \tag{24}
\end{equation*}
$$

Several remarks are due in order to justify the intuitive appeal of the new definition of rain rate or intensity via equation (22), and its consequence (24) relating the new with the classical notion of rain rate.

First of all the new notion of rain rate is sensibly related with the classical one. More specifically, equations (15) and (24) show that the randomness of both $R$ and $t_{0}$ stems from the same random factor $\mathrm{N}(\Delta \mathrm{t})$ (i.e. total number of raindrops collected during $\Delta t$ ), and both are proportional to the third moment of droplet diameter. Moreover, (24) implies that the product $\mathrm{R} \cdot \mathrm{t}_{0}$ remains constant, meaning that a fixed volume of water $V(\Delta t)$ accumulated during a fixed length of time $\Delta t$, has occurred from more intense rainfall when the mean volume of its droplet population is small, and from milder rainfall when the mean droplet volume is large. This is indeed reasonable, in the sense that the population with droplets of small mean volume will consist of more $\mathrm{N}(\Delta \mathrm{t})$ drops, and therefore the denominator of (22) will be larger than in the case of a population with large mean volume. This in turn implies that the ratio $t_{0}$ with fixed numerator $\Delta t$ in (22) will be smaller.

But shorter interarrival time $\mathrm{t}_{0}$ quantifies exactly what is meant by more intense rain !!!

Second, if identical rainfall heights were accumulated in two different raingauges, measuring rainfall events with identical probability distributions of droplet size, during the same time interval, then the rainfall measured by the gauge with the larger orifice size must have been more intense, corresponding naturally to smaller $\mathrm{t}_{0}$.

Third, for fixed mean droplet volume (i.e. fixed probability distribution of drop diameter size), fixed $\Delta t$, and fixed orifice size $\alpha$, larger height of accumulation $H(\Delta t)$ implies smaller $t_{0}$, and thus more intense rain as well.

Fourth, for fixed mean droplet volume, fixed $\Delta \mathrm{t}$, and fixed orifice size $\alpha$, larger rain rate R implies also smaller $\mathrm{t}_{0}$ and thus more intense rain.

The above remarks are direct mathematical consequences of equation (24), showing good and sensible agreement between the intuitive implications of both the classical notion of rain rate $R$ and the new one introduced via $t_{0}$.

## 4. DETECTABILITY OF RAIN DROPLETS BY RADAR

In this section we shall try to give a heuristic argument on the basis of which one may calculate explicitly a coefficient of detectability of a single rain-drop via a radar-beam scanning the column of air just near the orifice of a rain-gauge.

### 4.1 Empty Space Ratio

Continuing our line of thought about the idealised formation of single droplet arrivals at the orifice area, each with fixed uniform terminal velocity $u_{0}$ and interarrival time
$t_{0}$, it is clear that during any time interval of length $t_{0}$ there is only a single droplet in the cylindrical air column of height

$$
\begin{equation*}
h_{0}=u_{0} \cdot t_{0} \tag{25}
\end{equation*}
$$

just above the measuring gauge, falling towards the orifice of the gauge. That is, $h_{0}$ can be thought of as being the average height difference between two rain drops which arrive consecutively at the gauge's orifice area, both falling with terminal velocity $u_{0}$. This implies that the air column of volume

$$
\begin{equation*}
V_{c}=a \cdot h_{0}=a \cdot u_{0} \cdot t_{0} \tag{26}
\end{equation*}
$$

just above the orifice area of the gauge contains only one single droplet of volume $V_{0}$, and the rest of its space is empty of droplets. Thus, it is natural to define a dimensionless quantity which we shall refer to as the empty space ratio

$$
\begin{equation*}
q=\frac{V_{c}-V_{0}}{V_{c}}=1-\frac{V_{0}}{V_{c}} \tag{27}
\end{equation*}
$$

which after some elementary algebra via a series of substitutions from previously derived formulae reduces to

$$
\begin{equation*}
q=1-\frac{R}{u_{0}} \tag{28}
\end{equation*}
$$

From equation (27) it is clear that $0 \leq q \leq 1$ for every rain event, where the value $q=1$ corresponds to the case of no rain, and the value $q=0$ corresponds to the case of extremely intense rainfall so that every bit of the column just above the orifice is occupied by falling droplets. That is, the empty space ratio during a rain event of classical rain rate $R$, is equal to 1 minus the ratio of $R$ over the mean terminal velocity of the population of falling droplets. It is interesting to note that since $R$ and $u_{0}$ are both expressed in units of length over time $[\mathrm{L} / \mathrm{T}], \mathrm{q}$ is indeed a unitless quantity, expectedly much smaller than 1 , since the average rain rate expresses a much smaller "speed" than the mean terminal velocity $u_{0}$ of the free-falling droplets. Equation (28) and its equivalent

$$
\begin{equation*}
R=u_{0} \cdot(1-q) \tag{29}
\end{equation*}
$$

establish a relation between classical rain rate $R$, the average terminal velocity $u_{0}$ of falling droplets, and the empty space ratio $q$. This suggests yet another possibility of defining rainfall intensity via the empty space ratio, so that intense rain corresponds to low values of the empty space ratio, and mild rainfall corresponds to values near the upper bound 1 of the empty space ratio.
Moreover, equation (29) reveals that during any rain event the intensity of rain cannot exceed the upper bound defined by the average terminal velocity $u_{0}$ of falling droplets. This is a rather remarkable point which can be useful in further modelling of the probability distribution of rain rate not on the unbounded interval $[0,+\infty$ ) as usual(Kedem and Pavlopoulos 1991, Pavlopoulos and Kedem 1992), but on the finite interval $\left[0, u_{0}\right.$ ) or $\left[0, u_{0}\right]$ instead.

### 4.2 Coefficient of Detectability

Let us now suppose that a precipitation radar has been installed at certain distance from a fixed rain gauge. Using the notions defined earlier, and the notion of detectability as defined later in this subsection, we shall heuristically argue so as to reveal some limitations of the radar's ability to detect a population of falling droplets.

Radar beams are projected with an inclination of $1^{\circ}$ from the ground, so as to avoid interference of any echoes from the ground. The width of a radar beam depends on the type of radar, but it is usually about $1^{\circ}$ as well. Therefore, we may assume that at any given instant the scanning radar beam has a cone-like shape, with an opening angle of $1^{\circ}$, whose axis of symmetry is also tilted by an angle of $1^{\circ}$ from the ground. This geometric setting produces some problems as the distance of target from the radar increases, because the beam scanned area also increases in such a way that the intensity of reflection decreases. That is, in long distances, light rain may be missed due to the weakness of the reflected radiation, or it may even be totally masked by very strong rainfall reached before it, in shorter distance from the radar. Another problem is that due to the inclination of the beam, and because the globe is curved, at longer distance from the radar the beam hits higher levels of the atmosphere. Mainly for these reasons, the use of radars for detection of rainfall is limited to distances within which they are still effective.

In order to overcome concerns of those types, in the following calculations we consider a very tiny beam, that is, a very small part of the entire $1^{\circ}$ beam mentioned earlier. For such a beam one may further assume that within a certain range from the location where the radar is installed, it moves horizontally, and also does not diverge, so that the intensity of its radiation remains rather uniform, as long as it does not hit any obstacles on its way.

A new dimensionless quantity $Q$, referred to as coefficient of detectability of raindrops, is defined by the ratio

$$
\begin{equation*}
Q=\frac{A_{0}}{A_{c}} \tag{30}
\end{equation*}
$$

where $A_{0}$ is the area covered by the horizontal projection of a falling droplet of spherical volume $V_{0}$ and terminal velocity $u_{0}$, and $A_{c}$ is the area covered by the horizontal projection of the cylindrical volume $V_{c}$, along the direction of the radar's scanning beam.

The coefficient of detectability may be interpreted as being a geometric probability of the event that an average rain droplet is detected by a tiny radar beam scanning horizontally the space just above the orifice of a fixed rain-gauge. That space is defined to be a cylinder of height $h_{0}$, with a circular base of area $a$ equal to the orifice size. From a frequentist's point of view, assuming that detections of average droplets in a series of such non-overlapping cylinders, falling one after the other towards the orifice of the gauge, form a Bernoulli process with probability of success $Q$, then the reciprocal coefficient of detectability $1 / Q$ may be interpreted as the least number of cylinders needed to be scanned, so that expectedly at least one of them will be detected to contain a droplet. Recall that each cylinder of height $h_{0}$, and with circular base of area $a$ equal to the orifice size, by definition contains exactly
one average droplet falling with terminal velocity $u_{0}$, which is either detected with probability $Q$ or is missed with probability $1-Q$ by the horizontally scanning beam.

In order to calculate the coefficient of detectability $Q$, we first note that the spherical volume $V_{0}$ has radius of length $(1 / 2) \cdot \sqrt[3]{E\left(D^{3}\right)}$, and therefore its horizontal projection is just a circular disk with area

$$
\begin{equation*}
A_{0}=\frac{\pi}{4} \cdot\left\{E\left(D^{3}\right)\right\}^{2} \tag{31}
\end{equation*}
$$

On the other hand, the horizontal projection of the cylindrical volume $V_{c}$ is a rectangle of area

$$
\begin{equation*}
A_{c}=2 h_{0} \sqrt{\frac{a}{\pi}}=2 u_{0} t_{0} \sqrt{\frac{a}{\pi}}=\frac{1}{3} \sqrt{\frac{\pi}{a}} \cdot E\left(D^{3}\right) \cdot \frac{u_{0}}{R} \tag{32}
\end{equation*}
$$

Thus, dividing (31) by (32) one finds the coefficient of detectability

$$
\begin{equation*}
Q=\frac{3 \sqrt{a \pi}}{4\left\{E\left(D^{3}\right)\right\}^{1 / 3}} \cdot \frac{R}{u_{0}} \tag{33}
\end{equation*}
$$

which in light of equation (29) is also related with the empty space ratio $q$ by

$$
\begin{equation*}
Q=\frac{3 \sqrt{a \pi}}{4\left\{E\left(D^{3}\right)\right\}^{1 / 3}} \cdot(1-q) \tag{34}
\end{equation*}
$$

For the reasons mentioned earlier, the quantity $t_{0} / Q$ is a lower bound of the length of scanning time until expectedly at least one average droplet shall be detected by the (tiny) horizontal beam, just above the orifice of the gauge. Taking this very line of thought one step further, from the temporal frame over the orifice of a fixed gauge, to the spatial frame at a fixed instant of time, the quantity $a / Q$ is a lower bound of the size of area covered by a dense array of gauges, over which simultaneous horizontal scanning by parallel radar beams shall detect expectedly at least one average droplet.

## 5. DISCUSSION AND CONCLUSIONS

The approach we adopted in the present study relies on the understanding that rainfall is not a continuous process, neither in space (not discussed in this study), nor in time. Instead, rainfall is manifested as being a discrete process in both space and time. Once adopted, this approach excludes the use of techniques or terms based on continuity in order to describe a rain event. Thus, for example, the spatial distribution of a single storm should not be represented by systems of continuous contours, known as isohyetal curves [Kutiel and Kay (1994), Kay and Kutiel (1996)].

Likewise, the use of units of length over time $[\mathrm{L} / \mathrm{T}]$, such as $\mathrm{mm} / \mathrm{hr}$, in order to measure instantaneous rainfall intensity is also inappropriate. In fact, any measure
according to which a unit of length (e.g. mm, Km, etc.) is divided by a unit of time (e.g. min, hr, etc.), implies measurement of speed or velocity, which is a physical variable continuous with respect to time. As much as units like $\mathrm{Km} / \mathrm{hr}$ are appropriate for measurement of speed of a running vehicle, they should not be used to describe rain rate. For vehicles, there is a physical constraint according to which, when the vehicle accelerates, say from $60 \mathrm{Km} / \mathrm{hr}$ to $90 \mathrm{Km} / \mathrm{hr}$, it has to run exhaustively at all instantaneous speeds between $60 \mathrm{Km} / \mathrm{hr}$ and $90 \mathrm{Km} / \mathrm{hr}$, without skipping any subinterval of them, say between $70 \mathrm{Km} / \mathrm{hr}$ and $80 \mathrm{Km} / \mathrm{hr}$. On the other hand, this physical constraint of continuity is no longer valid in the case of statistical populations of falling raindrops, manifesting a discrete process with respect to time.

According to our new notion of rain rate, as being the elapsed time between the entering of two consecutive raindrops into a measuring raingauge, the intensity of rainfall may increase, say from 0.75 sec to 0.15 sec , at a certain location and during the same rain event, without necessarily being forced to reach all the intermediate values, and this is another manifestation of the discontinuous nature of rainfall !!!

This shows that the use of units such as $\mathrm{mm} / \mathrm{hr}$ for the measurement of rainfall intensity may convey a misleading understanding about the physical variable intended to be measured. However, equation (24) reconciles a relationship between the new and the classical notions of rain rate, in the sense that one may still use the classical notion of rain rate in a merely conventional way, accounting also for the statistical properties of the population of falling raindrops. These statistical properties have been summarised in this article simply by a discrete probability distribution with finite third moment, governing the limiting relative frequencies of the random diameter sizes of falling raindrops. The task of more detailed modelling of such candidate probability distributions of diameter size is a much deeper one, which should be pursued later on in a sequel article.

Nevertheless, we believe that the proposed new approach is quite appealing from the viewpoint of its possible applications. For example, in Section 4 we demonstrated a way of assessing quantitatively the ability $Q$ of an ideal Radar beam to detect falling raindrops, and its limitations in time and space. Another interesting application may be the quantitative assessment of geomorphic consequences resulting from erosional effects of rainfall, which are highly dependent on both the instantaneous rate of rainfall-runoff and the rate of infiltration.

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