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THE FOLDED t DISTRIBUTION

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ABSTRACT

Measurements are frequently recorded without their algebraic sign. As a consequence the underlying distribution of measurements is replaced by a distribution of absolute measurements. When the underlying distribution is t the resulting distribution is called the "*folded-t distribution*". Here we study this distribution, we find the relationship between the folded-t distribution and a special case of the folded normal distribution and we derive relationships of the folded-t distribution to other distributions

pertaining to computer generation. Also tables are presented which give areas of the folded-t distribution.

1. INTRODUCTION

Very often, the process of recording a measurement represented by a random variable X can be thought of as resembling a distortion mechanism that leads to the random variable $Y=|X|$. In other words, the algebraic sign of the measurement is disregarded by the recording mechanism and as a result the observed distribution of measurements is the distribution of absolute measurements. This distortion process is known in the literature as folding as it can be regarded as leading to a distribution that would result by folding the probability mass associated with the original random variable around the origin. The folded normal distribution of Leone et al (1961) constitutes one such example of folded distribution which arises in industrial practice in connection with measuring the flatness or straightness of objects. So, instead of the actual $N(\mu,\sigma^2)$ distribution of measurements one observes what is termed the folded normal distribution with parameters

$$\mu_f = \sqrt{2/\pi}\,\sigma\, e^{-\mu^2/2\sigma^2} + \mu(1-2\Phi(-\mu/\sigma))$$

and

$$\sigma_f^2 = \mu^2+\sigma^2-\mu_f^2$$

where $\Phi(x)$ denotes the distribution function of the standard normal distribution.

A natural question would be to investigate situations that would imply folding of the t distribution, another distribution that is widely used in statistics.

In this paper a practical problem in the area of validating forecasting models is given that motivates the definition of the folded t distribution which is subsequently studied. In particular, the next sections describe the practical situation in which a folded t distribution may be observed (section 2), provide its probability density function, mean and variance (section 3) and demonstrate the relationship between the folded t distribution and the folded standard normal distribution (section 4)and the folded t distribution with other distributions (section 5). Finally, an algorithm is used for the determination of the values of the distribution function of the folded t distribution and extracts of the obtained tables are given for certain values of the parameters.

2. SCORING MODEL BEHAVIOUR AND THE FOLDED t DISTRIBUTION

Let

$$Y_t = X_t\beta + e_t$$

be a linear model where Y_t is an $l_t \times 1$ vector of values of the dependent random variable, X_t is an $l_t \times m$ matrix of known coefficients with $l_t \geq m$ and $|X_t' X_t| \neq 0$, β is an $m\times 1$ vector of regression coefficients and e_t is an $l_t \times 1$ vector of normal error random variables with mean $E(e_t)=0$ and $V(e_t)=\sigma^2\, I_t$. Here I_t is an $l_t \times l_t$ indentity matrix. Using this model the value of the dependent random variable can be predicted for the (t+1)-th year by $\hat{Y}^0_{t+1} = X^0_{t+1}\hat{\beta}_t$ where for t=0,1,2,... $\hat{\beta}_t$ is the least square estimator of β at time t+1 as given by

$$\hat{\beta}_t = \left[X_t' X_t\right]^{-1} X_t Y_t$$

and X^o_{t+1} is a 1×m vector of values of the regressors for the (t+1)-th year. Xekalaki and Katti (1984) introduced a sequential scheme to evaluate the forecasting potential of such models which amounted to scoring the "forecasting performance" of the model at each of a number of points in time and obtaining a final rating based on them. The "forecasting performance" is a term used in the present paper to refer to the closeness of the predicted value $\hat{Y}^o_{t+1}$ to the actually observed value Y^o_{t+1} as reflected by the various scores that can be used. So, a set of n predicted values along with the corresponding actual values lead to a sequence of n scores that can be used in any meaningful manner to produce a final rating of the forecasting potential of the model. Motivated by Xekalaki and Katti's (1984) scheme one may naturally consider using a score that will reflect how far from the actual value Y^o_{t+1} the predicted value $\hat{Y}^o_{t+1}$ is relative to the observed standard deviation of the values of the dependent random variable. This is equivalent to using

$$r_{t+1} = \frac{|\hat{Y}^o_{t+1} - Y^o_{t+1}|}{S_t \sqrt{1 + X^o_{t+1}(X'_t X_t)^{-1} X'^o_{t+1}}}, \quad t=0,1,2,\ldots \qquad (2.1)$$

as a score of the forecasting performance of the model at time t where S^2_t is the usual estimator of σ^2_t given by

$$S^2_t = \left[Y_t - X_t\hat{\beta}_t\right]' \left[Y_t - X_t\hat{\beta}_t\right] / (l_t - m).$$

Obviously, (2.1) disregards the direction in which $\hat{Y}^o_{t+1}$ differs from Y^o_{t+1}. This can occasionally be of some importance to the decision maker since a consistently unbalanced proportion of overestimated (or underestimated) y values may indicate an inherent inadequacy of the used model. Hence one may look upon

(2.1) as a distorted reflection of the forecasting performance of the model at time t and consider the resulting distributions of the scores as folded versions of the distributions of the standardized differences $Y^o_{t+1}-\hat{Y}^o_{t+1}$, t=0,1,2,.... These standardized differences are, by the assumptions of the model, independently t distributed with l_t-m degrees of freedom as pointed out by Psarakis and Panaretos (1990). Therefore, studying the distribution of the random variable $|T|$ when T has a t distribution with ν degrees of freedom is in order and in the sequel we will refer to it as the **folded t distribution with ν degrees of freedom.**

3. FOLDING A t DISTRIBUTION WITH ν DEGREES OF FREEDOM.

Let T be a random variable having the t distribution with ν degrees of freedom as defined by the density function

$$f_T(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad \nu\in N,\ x\in R \qquad (3.1)$$

and let $F_T(x)$ denote its distribution function. Folding (3.1) will result in the distribution of the random variable $W=|T|$ and it is evident that the distributions of W and T will relate thus

$F_W(x) = F_T(x)-F_T(-x)$, x>0.

Due to the symmetricity of (3.1) around the origin this relationship reduces to

$$F_W(x) = 2F_T(x)-1, \qquad x>0 \qquad (3.2)$$

which, in terms of probability density functions is equivalent to

$f_W(x) = 2f_T(x), \qquad x>0.$

This leads to the following definition .

Definition 3.1 A non-negative real valued random variable X will be said to have the folded t distribution with ν degrees of freedom if its probability density function is given by

$$f_X(x) = \frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad x>0, \quad \nu\in N. \tag{3.3}$$

Figure 1 provides a graph of the probability density function of the folded t-distribution for some values of the parameter ν.

Theorem 3.1 Let X be a non-negative real valued randomvariable having the folded t distribution with ν degrees of freedom. Then

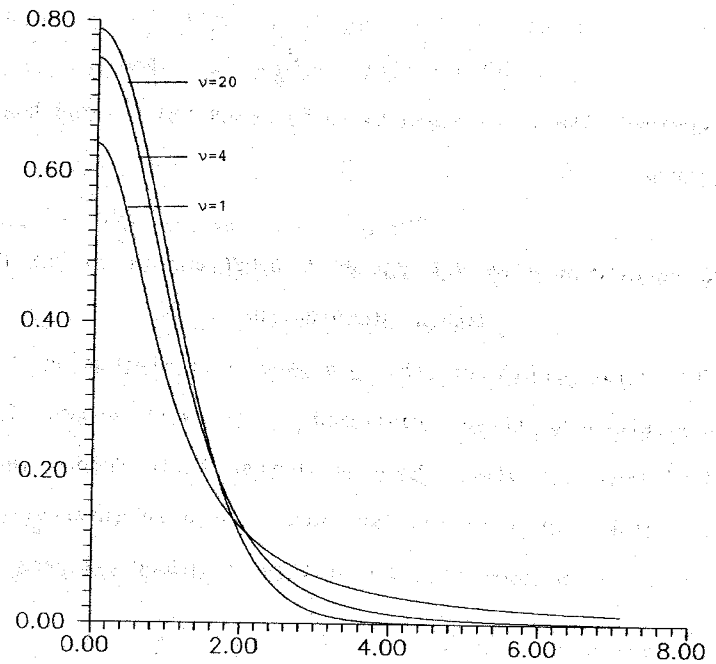
$$\text{(i)} \quad E(X) = \begin{cases} 2\sqrt{\frac{\nu}{\pi}}\,\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu-1)}, & \nu>1 \\ \infty, & \nu=1 \end{cases} \tag{3.4}$$

$$\text{(ii)} \quad V(X) = \begin{cases} \dfrac{\nu}{\nu-2} - \dfrac{4\nu}{\pi(\nu-1)^2}\left(\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right)^2, & \nu>2 \\ \infty, & \nu\le 2 \end{cases} \tag{3.5}$$

Proof

(i) We have

$$E(X) = \frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}}\int_0^\infty x\left(1+\frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} dx$$



Figure–1

Probability density function
of the folded–t distribution

$$= \frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}} \int_1^{\infty} x^{-\frac{1}{2}(\nu+1)}\, dx.$$

Then E(X) does not exist for $\nu=1$ while for $\nu>1$ E(X) is given by (3.4).

(ii) Since $X=|T|$, where T is some t random variable we have for the variance that

$$V(X) = E(X^2) - \left[E(X)\right]^2 = E(T^2) - \left[E(T)\right]^2. \quad (3.6)$$

But

$$E(T^2) = V(T) = \frac{\nu}{\nu-2}, \qquad \nu>2.$$

This combined with (3.6) leads to (3.5) Hence the theorem has been established.

4. THE RELATIONSHIP OF THE FOLDED t DISTRIBUTION TO THE FOLDED NORMAL DISTRIBUTION.

The relationship of the ordinary t distribution to the ordinary standard normal distribution is well known. From a practical point of view, checking whether their folded versions exhibit relationships of similar nature would be interesting. In the sequel, it is demonstrated that this is indeed the case.

Let X be a non negative real valued random variable having the folded normal distribution with probability density function given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\left\{e^{-(x-\mu)^2/2\sigma^2}+e^{-(x+\mu)^2/2\sigma^2}\right\}, \quad x>0. \tag{4.1}$$

(For the definition, properties, estimation problems and applications of the folded normal distribution see Leone et.al (1961), Elandt (1961), Johnson (1962), Gilbert and Mosteller (1966), Risvi (1971), Sundberg (1974), Nelson (1980), and Sinha (1983)).

Letting $\mu=0$ and $\sigma^2=1$, (4.1) leads to the folded version of the standard normal distribution which in the sequel we refer to as the **folded standard normal distribution**.

Denote by $\Phi_f(x)$ and $\phi_f(x)$ the distribution function and the probability density function of this distribution.

Then

$$\phi_f(x)=\sqrt{\frac{2}{\pi}}e^{-x^2/2} \qquad , \quad x>0 \tag{4.2}$$

with mean $\mu=\sqrt{2/\pi}$ and variance $\sigma^2=1-\sqrt{2/\pi}$.

Theorem 4.1 Let Z and X be two non-negative real-valued random variables with distributions defined by the probability density functions given by (4.1) and (3.3) respectively. Let $\Phi_f(z)$ be defined as above and denote the distribution function of X by $F_\nu(x)$. Then

$$\lim_{\nu\to\infty} F_\nu(x) = \Phi_f(x) \qquad , \quad x>0. \tag{4.3}$$

Proof: Let $\Phi(x)$ and $F_{t_\nu}(x)$ denote the distribution functions of the ordinary standard normal distribution and the ordinary t distribution with ν degrees of freedom. Then, (4.1) suggests that

$$\Phi_f(x) = 2\,\Phi(x)-1 \ . \tag{4.7}$$

Then, using (3.2)

$$\lim_{\nu\to\infty} F_\nu(x) = 2\lim_{\nu\to\infty} F_{t_\nu}(x)-1$$
$$= 2\Phi(x)-1$$

Hence the theorem has been established.

It has therefore been shown that the folded standard normal distribution is a limiting form of the folded t distribution when the number of degrees of freedom becomes very large, just like the ordinary standard normal is a limiting form of the ordinary t distribution.

5. SOME RELATIONSHIPS OF THE FOLDED t DISTRIBUTIONS WITH OTHER DISTRIBUTIONS PERTAINING TO COMPUTER GENERATION.

In this section the interrelationships of the folded t distribution to various existing distributions will be indicated with emphasis on those which may potentially be utilized in the generation of random numbers. It turns out that the folded t distribution relates to well known distributions such as the F-distribution, the χ^2- distribution, the beta distribution of the second type, and the folded normal distribution as well as the standard and half-Cauchy distributions as may be shown by the following theorems.

Theorem 5.1. Let Z be a non-negative real valued random variable having the folded standard normal distribution. Let X be another non-negative real valued random variable distributed independently of Z according to a χ^2 distribution with ν degrees of freedom. Then the random variable $T=Z/\sqrt{X/\nu}$ has the folded t distribution.

Proof: It is well known that if Z_0 is an ordinary standard normal random variable independent of X then the random variable $T_0=Z_0/\sqrt{X/\nu}$ is an ordinary t random variable with ν degrees of freedom. Then

$$T=Z/\sqrt{X/\nu} \overset{d}{=} |Z_0|/\sqrt{X/\nu} = |T_0|.$$

Hence the result.

Theorem 5.2. Let X, Y be independent non-negative, real valued random variables having the chi-distribution with parameters 1 and

ν, respectively, and probability density functions given by

$$f_X(x) = \frac{2^{1/2}}{\Gamma(1/2)} e^{-x^2/2} \quad ,x>0 \qquad (5.2)$$

and

$$f_Y(y) = \frac{1}{2^{\frac{\nu}{2}-1}\,\Gamma(\nu/2)} y^{\nu-1} e^{-y^2/2}, \quad y>0 \qquad (5.3)$$

respectively. Then, the random variable

$$Z = \nu^{1/2} X/Y$$

has the folded t distribution with ν degrees of freedom.

Proof: It is known that the standard folded normal distribution coincides with the chi-distribution with 1 degree of freedom. Further if W has the chi-square distribution with ν degrees of freedom then $Y \overset{d}{=} \sqrt{W}$. Hence the result follows from theorem 5.1.

Theorem 5.3 Let Z_1, Z_2 be two independent non-negative, real-valued random variables having the folded standard normal distribution. Then

$$Y=Z_1/Z_2$$

has the folded t distribution with 1 degree of freedom.

As mentioned above the folded standard normal distribution is a special case of the chi-distribution. Therefore, theorem 5.3 leads to the following theorem:

Theorem 5.4 Let Z_1, Z_2 be independent chi-variables with parameter 1 and probability density function as given by (5.2). Then

$$Y=Z_1/Z_2$$

has the folded t distribution with 1 degree of freedom.

Theorem 5.5 Let X be a folded t variable with probability density function given by (3.3). Let Y be another random variable having the standard half-Cauchy distribution with probability density function given by

$$f_Y(y) = \frac{2}{\pi(1+x^2)}, \quad x > 0.$$

Then

$$X \overset{d}{=} Y,$$

i.e. the folded t distribution with ν=1 degrees freedom coincides with the half- Cauchy distribution.

Theorem 5.6 Let X be defined as in theorem 5.5. Then

$$Y = X^2$$

has the F distribution with 1 and ν degrees of freedom and probability density function given by

$$f(x) = \frac{1}{B(\frac{1}{2}, \frac{\nu}{2})} \nu^{\nu/2} x^{-1/2} (\nu+x)^{-\frac{1+\nu}{2}}$$

Theorem 5.7 Let X be defined as theorem 5.5. Then

$$Y = X^2/\nu$$

has the beta distribution of the second type (beta prime or Pearson type VI distribution) with parameters 1/2 and $\nu/2$ and probability density function given by

$$f(x) = \frac{1}{B(\frac{1}{2}, \frac{\nu}{2})} x^{-\frac{1}{2}} (1+x)^{-\frac{\nu+1}{2}}, \quad x > 0.$$

Theorem 5.8 Let X be an F variable with ν and ν degrees of freedom and probability density function given by

$$f_X(x) = \frac{1}{B(\frac{\nu}{2}, \frac{\nu}{2})} x^{\nu/2-1} (1+x)^{-\nu}, \qquad x>0.$$

Then

$$Y = \frac{\nu^{1/2}}{2}|X^{1/2}-X^{-1/2}|$$

has the folded t distribution with ν degrees of freedom.

Theorem 5.9 Let X_1, X_2 be i.i.d. random variables each having the χ^2-distribution with ν degrees of freedom and probability density function given by

$$f(x) = \frac{2^{-\nu/2}}{\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \qquad x>0.$$

Then

$$Y = \frac{\nu^{1/2}}{2} \frac{|X_1-X_2|}{(X_1X_2)^{1/2}}$$

has the folded-t distribution with ν degrees of freedom.

Theorem 5.10 Let X be a random variable having the generalized Cauchy distribution with parameters α, $\beta=1$, $\gamma=2$ and $\delta=(\nu+1)/2$ and probability density function given by

$$f_X(x)=\frac{1}{B(1/2,\nu-1/2)} [1+(x-\alpha)^2]^{-(\nu+1)/2}, \quad x,\alpha\in R.$$

Then

$$Y = \nu^{1/2}|x-\alpha|$$

has the folded-t distribution with $\frac{\nu+1}{2}$ degrees of freedom.

6. AREAS OF THE FOLDED t DISTRIBUTION

In connection with practical problems one may need to determine the probabilities associated with the folded t distribution.

Obviously, for small ν one can always appeal to the existing tables of the ordinary t distribution and convert the tabulated areas using (3.2).On the other hand, for large ν Leone et al's (1961) tables of the folded standard normal distribution can be utilized to provide limiting values of the areas wished to be determined as implied by theorem 4.1.

The evaluation of integrals of the folded t distribution of the type

$$F_{\nu}(t) = \int_0^t \frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}} \left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \quad , \qquad t>0.$$

can be achieved by a slight modification of Cooper's (1968) algorithm for the areas of the ordinary t distribution (see also Griffiths and Hill (1985)). Extracts of the tables generated are provided in Appendices I and II.

APPENDIX I

Areas of the Folded-t Distribution

$$F_t^f\,(t_f) = \int_0^{t_f} f_t^f(x)dx$$

t_f \ v	1	2	3	4	5	10	20
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0634	0.0705	0.0733	0.0748	0.0758	0.0777	0.0786
0.2	0.1256	0.1400	0.1457	0.1487	0.1506	0.1545	0.1565
0.3	0.1855	0.2075	0.2162	0.2209	0.2237	0.2296	0.2327
0.4	0.2422	0.2721	0.2840	0.2904	0.2943	0.3024	0.3066
0.5	0.2951	0.3333	0.3485	0.3566	0.3617	0.3721	0.3774
0.6	0.3440	0.3905	0.4092	0.4191	0.4253	0.4381	0.4447
0.7	0.3888	0.4436	0.4656	0.4775	0.4848	0.5001	0.5080
0.8	0.4295	0.4923	0.5178	0.5314	0.5399	0.5376	0.5669
0.9	0.4665	0.5369	0.5655	0.5810	0.5906	0.6107	0.6211
1.0	0.5000	0.5773	0.6090	0.6261	0.6367	0.6591	0.6707
1.1	0.5303	0.6139	0.6483	0.6669	0.6785	0.7029	0.7156
1.2	0.5577	0.6470	0.6837	0.7036	0.7161	0.7422	0.7558
1.3	0.5825	0.6767	0.7155	0.7365	0.7497	0.7772	0.7916
1.4	0.6051	0.7035	0.7440	0.7659	0.7796	0.8082	0.8231
1.5	0.6256	0.7276	0.7694	0.7920	0.8061	0.8355	0.8507
1.6	0.6443	0.7492	0.7921	0.8151	0.8295	0.8593	0.8747
1.7	0.6615	0.7687	0.8123	0.8356	0.8501	0.8800	0.8953
1.8	0.6771	0.7863	0.8303	0.8537	0.8682	0.8979	0.9130
1.9	0.6915	0.8021	0.8463	0.8697	0.8841	0.9133	0.9280
2.0	0.7048	0.8165	0.8607	0.8838	0.8980	0.9266	0.9407
2.1	0.7170	0.8294	0.8734	0.8963	0.9102	0.9379	0.9513
2.2	0.7284	0.8412	0.8848	0.9073	0.9209	0.9475	0.9602
2.3	0.7390	0.8518	0.8550	0.9170	0.9302	0.9557	0.9676
2.4	0.7486	0.8615	0.9041	0.9256	0.9383	0.9626	0.9737
2.5	0.7577	0.8704	0.9123	0.9332	0.9455	0.9685	0.9786
2.6	0.7662	0.8784	0.9196	0.9399	0.9517	0.9735	0.9828
2.7	0.7742	0.8858	0.9262	0.9459	0.9572	0.9776	0.9862
2.8	0.7816	0.8926	0.9321	0.9511	0.9620	0.9812	0.9889
2.9	0.7886	0.8988	0.9374	0.9558	0.9662	0.9841	0.9911
3.0	0.7951	0.9045	0.9423	0.9600	0.9699	0.9866	0.9929
3.1	0.8013	0.9098	0.9467	0.9637	0.9731	0.9887	0.9943
3.2	0.8071	0.9146	0.9506	0.9671	0.9760	0.9905	0.9955
3.3	0.8127	0.9191	0.9542	0.9700	0.9785	0.9919	0.9964
3.4	0.8180	0.9233	0.9575	0.9727	0.9807	0.9932	0.9971
3.5	0.8228	0.9271	0.9605	0.9751	0.9827	0.9942	0.9977

(continued)

Areas of the Folded-t Distribution (contd).

t_f \ v	1	2	3	4	5	10	20
3.6	0.8275	0.9307	0.9632	0.9772	0.9844	0.9951	0.9982
3.7	0.8319	0.9341	0.9657	0.9791	0.9860	0.9959	0.9985
3.8	0.8362	0.9372	0.9680	0.9801	0.9873	0.9965	0.9988
3.9	0.8402	0.9401	0.9700	0.9824	0.9885	0.9970	0.9991
4.0	0.8440	0.9428	0.9720	0.9838	0.9896	0.9974	0.9993
4.1	0.8477	0.9453	0.9737	0.9851	0.9906	0.9978	0.9994
4.2	0.8512	0.9477	0.9753	0.9863	0.9915	0.9981	0.9995
4.3	0.8545	0.9499	0.9768	0.9873	0.9922	0.9984	0.9996
4.4	0.8577	0.9520	0.9782	0.9883	0.9929	0.9986	0.9997
4.5	0.8608	0.9539	0.9795	0.9891	0.9936	0.9988	0.9998
4.6	0.8637	0.9558	0.9806	0.9899	0.9941	0.9990	0.9998
4.7	0.8665	0.9575	0.9817	0.9907	0.9946	0.9991	0.9998
4.8	0.8692	0.9592	0.9828	0.9913	0.9951	0.9992	0.9998
4.9	0.8718	0.9608	0.9837	0.9919	0.9955	0.9993	0.9999
5.0	0.8743	0.9622	0.9846	0.9925	0.9958	0.9994	0.9999
5.1	0.8767	0.9636	0.9854	0.9930	0.9962	0.9995	0.9999
5.2	0.8790	0.9649	0.9861	0.9934	0.9965	0.9996	0.9999
5.3	0.8812	0.9662	0.9867	0.9939	0.9968	0.9996	0.9999
5.4	0.8834	0.9673	0.9875	0.9943	0.9970	0.9997	0.9999
5.5	0.8855	0.9685	0.9881	0.9946	0.9972	0.9997	0.9999
5.6	0.8875	0.9695	0.9887	0.9950	0.9974	0.9997	0.9999
5.7	0.8894	0.9705	0.9893	0.9953	0.9976	0.9998	0.9999
5.8	0.8913	0.9715	0.9898	0.9956	0.9978	0.9998	0.9999
5.9	0.8931	0.9724	0.9902	0.9958	0.9980	0.9998	0.9999
6.0	0.8948	0.9733	0.9907	0.9961	0.9981	0.9998	0.9999

APPENDIX II

Critical values of the Folded-t Distribution

v	α=0.55	α=0.60	α=0.70	α=0.75	α=0.80	α=0.90	α=0.95	α=0.975	α=0.99	α=0.995
1	1.1708	1.3764	1.9626	2.4142	3.0777	6.3137	12.7062	25.4517	63.6567	127.3215
2	0.9313	1.0606	1.3862	1.6036	1.8856	2.9199	4.3026	6.2053	9.9248	14.0895
3	0.8661	0.9782	1.2498	1.4227	1.6380	2.3534	3.1824	4.1765	5.8409	7.4533
4	0.8361	0.9407	1.1895	1.3445	1.5770	2.1325	2.7774	3.4954	4.6041	5.5976
5	0.8188	0.9193	1.1557	1.3010	1.4761	2.0156	2.5714	3.1644	4.0321	4.7733
6	0.8076	0.9055	1.1341	1.2734	1.4400	1.9437	2.4476	2.9696	3.7084	4.3178
7	0.7997	0.8958	1.1191	1.2544	1.4151	1.8951	2.3653	2.8420	3.5003	4.0302
8	0.7939	0.8887	1.1081	1.2404	1.3970	1.8600	2.3066	2.7522	3.3562	3.8333
9	0.7894	0.8832	1.0997	1.2297	1.3832	1.8336	2.2630	2.6857	3.2506	3.6904
10	0.7859	0.8788	1.0930	1.2213	1.3724	1.8129	2.2287	2.6344	3.1700	3.5821
11	0.7830	0.8753	1.0876	1.2145	1.3636	1.7963	2.2015	2.5937	3.1065	3.4972
12	0.7806	0.8724	1.0832	1.2089	1.3564	1.7827	2.1794	2.5606	3.0552	3.4290
13	0.7786	0.8699	1.0795	1.2042	1.3504	1.7713	2.1609	2.5332	3.0130	3.3730
14	0.7769	0.8678	1.0763	1.2002	1.3452	1.7617	2.1453	2.5101	2.9774	3.3262
15	0.7754	0.8660	1.0735	1.1968	1.3408	1.7535	2.1320	2.4904	2.9473	3.2866
16	0.7741	0.8645	1.0711	1.1938	1.3369	1.7463	2.1204	2.4734	2.9213	3.2525
17	0.7730	0.8631	1.0690	1.1911	1.3334	1.7400	2.1103	2.4586	2.8988	3.2230
18	0.7720	0.8618	1.0672	1.1888	1.3306	1.7345	2.1014	2.4455	2.8790	3.1971
19	0.7711	0.8608	1.0655	1.1867	1.3279	1.7295	2.0935	2.4340	2.8615	3.1742
20	0.7702	0.8598	1.0640	1.1848	1.3255	1.7251	2.0865	2.4236	2.8459	3.1539
21	0.7695	0.8589	1.0627	1.1832	1.3234	1.7211	2.0801	2.4144	2.8319	3.1357
22	0.7689	0.8581	1.0614	1.1816	1.3214	1.7175	2.0744	2.4065	2.8193	3.1193
23	0.7682	0.8573	1.0603	1.1802	1.3196	1.7143	2.0691	2.3845	2.8078	3.1045
24	0.7677	0.8567	1.0593	1.1790	1.3180	1.7113	2.0644	2.3915	2.7974	3.0910
25	0.7672	0.8560	1.0584	1.1778	1.3165	1.7085	2.0600	2.3851	2.7875	3.0786
26	0.7667	0.8555	1.0575	1.1767	1.3151	1.7060	2.0560	2.3793	2.7792	3.0674
27	0.7663	0.8550	1.0567	1.1757	1.3139	1.7037	2.0523	2.3739	2.7712	3.0570
28	0.7659	0.8545	1.0560	1.1748	1.3127	1.7015	2.0490	2.3690	2.7635	3.0474
29	0.7655	0.8540	1.0553	1.1739	1.3116	1.6995	2.0457	2.3643	2.7568	3.0385
30	0.7652	0.8536	1.0546	1.1731	1.3106	1.6976	2.0427	2.3600	2.7504	3.0302
40	0.7626	0.8505	1.0500	1.1674	1.3032	1.6842	2.0215	2.3294	2.7049	2.9716
60	0.7601	0.8447	1.0454	1.1617	1.296	1.6710	2.0007	2.2995	2.6607	2.9149
120	0.7576	0.8444	1.0409	1.1560	1.2888	1.6580	1.9804	2.2703	2.6178	2.8602

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